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PREFACE.

THESE Notes on Plane Trigonometry have been in use for many years past to supplement the various text-books in the hands of my pupils. The work does not profess to be a complete treatise, but Chapters XIX. to the end contain all the more Advanced Propositions fully proved, and the book-work throughout is illustrated by the solution of more than two hundred problems.

The Student should make himself familiar with the definitions and the *proofs* of all the Propositions (marked PROP.); such proofs as are not given here may be found in any Elementary Trigonometry.

The Formulæ to which a number is attached should be committed to memory.

The Examination Papers are arranged in order, those of most recent date standing first; and I hope that such a complete set will be valuable to Teachers and Students, especially to those preparing for the Civil Service and other Examinations.

A. D. C.

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PLANE TRIGONOMETRY.

I.

Definitions, Units, and Measurements of Angles.

1. All measurements of lines, angles, etc., are referred to some fixed standard or *unit*, and **the measure of a quantity is the number of times which the quantity contains the unit.**

2. **Angles are measured in practice with reference to a right angle as the unit:** the right angle is divided into 90 equal parts, called *degrees*, or into 100 equal parts, called *grades*. These are again subdivided as follows :—

$60''$ (seconds)	$=1'$ (minute).	$100^{\backprime\backprime}$ (seconds)	$=1^{\backprime}$ (minute).
$60'$	$=1^\circ$ (degree).	$100^{\backprime}$	$=1^{g}$ (grade)
90°	$=$ a right angle.	100^{g}	$=$ a right angle.

The common measurement by degrees, minutes, etc., is called the **sexagesimal method.** The **centesimal**, or **French method**, is not practically used even in France.

3. **Ratio is the relation which one quantity bears to another in respect of magnitude, the comparison being made by considering what multiple part or parts the one quantity is of the other.**

All Ratios are Abstract Numbers.

4. **A commensurable number is one that can be expressed as an arithmetical fraction. An incommensurable number cannot be so expressed;** $\sqrt{2}=1{\cdot}41421\ldots$, $\sqrt{3}=1{\cdot}73205\ldots$, $\pi=3{\cdot}14159\ldots$, $\epsilon=2{\cdot}71828\ldots$, are all incommensurable.

5. A Polygon is a figure contained by any number of straight lines.

A *regular Polygon* has all its sides equal and all its angles equal.

6. The Perimeter of a figure is the sum of its sides.

Hence the perimeter of a regular polygon of n sides, each equal to a, is na.

7. Two diameters of a circle at right angles to each other form at their point of intersection four right angles (360°), and divide the circle into *Four Quadrants.*

8. *Extension of the Definition of an Angle.*

In Euclid, *a plane rectilineal angle* is defined to be "the inclination of two straight lines to one another, which meet together, but are not in the same straight line." No account is taken of the *direction* in which the inclination is measured, and the angles considered are in general less than two right angles.

In Eucl. III. 26, and VI. 33, there is no limit to the magnitudes of the angles.

In Trigonometry, we suppose an angle traced out by the revolution of one line from coincidence with another line about a fixed point, and the angle may be traced out by any number of revolutions of the line.

This line may revolve in the direction of the hands of a watch or in the opposite direction.

9. *Application of the signs* + *and* − *to indicate direction.*

All lines measured towards the right of a fixed point, called the origin, are considered positive.

All lines measured towards the left are considered negative.

All perpendicular lines measured upwards from a fixed horizontal line are considered positive, and those measured downwards are considered negative.

The revolving line is always considered positive.

Angles described by the revolution of a straight line in the same direction as the hands of a watch move are considered negative; angles described in the opposite direction are considered positive.

II.

Circular Measure.

1. Prop. The circumference of a circle varies as the radius.

i.e. The ratio of the circumference of a circle to its radius is the same for all circles.

$\therefore \dfrac{\text{Circumference}}{\text{Diameter}} =$ some constant quantity, which is denoted by π.

$\therefore$ Circumference $= \pi \times$ Diameter $= 2\pi r$, where r is the radius of the circle.

The value of π is $\dfrac{22}{7}$, or more correctly $\dfrac{355}{113} = 3{\cdot}14159 \ldots$

2. Prop. The angle subtended at the centre of a circle by an arc equal in length to the radius is invariable: *i.e.* the same for all circles.

This angle is the unit of circular measure, and its magnitude is $\dfrac{180^\circ}{\pi} = 57^\circ{\cdot}29577951 \ldots = 57^\circ\ 17'\ 44{\cdot}8'' = 206265'' = 63{\cdot}661977$ grades.

3. Def. **The circular measure of any angle is the number which expresses the ratio which the angle bears to the unit of circular measure, or, the circular measure of any angle at the centre of a circle is the ratio of the subtending arc to the radius of the circle.**

Angles expressed in circular measure are usually denoted by Greek letters.

By the angle whose circular measure is $\frac{3}{4}$, we mean that the angle is $\frac{3}{4}$ of the unit $= \frac{3}{4}$ of $\frac{180}{\pi}$ degrees; *or*, that the angle is subtended at the centre of a circle by an arc, which is $\frac{3}{4}$ of the radius of the circle, so that $\frac{\text{arc}}{\text{rad}} = \frac{3}{4}$.

The circular measure of 180° is π, of a right angle is $\frac{\pi}{2}$, of 1° is $\frac{\pi}{180}$, of $n°$ is $\frac{n\pi}{180}$, of $1'$ is $\frac{\pi}{180 \times 60}$, of any multiple of four right angles is $2n\pi$.

N.B.—The unit of circular measure is not π, and π is not $=180$.

4. Prop. Compare the number of degrees, grades, and units of circular measure in the same angle.

Let D, G, and C be the number of degrees, grades, and units of circular measure in any angle.

Then these bear the same ratio to a right angle.

$$\therefore \frac{D}{90} = \frac{G}{100} = \frac{C}{\frac{\pi}{2}} \qquad \text{(cxxviii. 1.)}$$

Equations connecting D, G, and C—

$$D = \frac{9}{10}G \text{ or } \frac{180C}{\pi}; \; G = \frac{10}{9}D \text{ or } \frac{200C}{\pi}; \; C = \frac{\pi D}{180} \text{ or } \frac{\pi G}{200}.$$

The equation connecting the English minutes (m) and French minutes (μ) in the same angle is—

$$\frac{m}{90 \times 60} = \frac{\mu}{100 \times 100}$$

$$\therefore 50m = 27\mu;$$

and if (s) be the English seconds and (σ) the French seconds,

$$\frac{s}{90 \times 60 \times 60} = \frac{\sigma}{100 \times 100 \times 100}$$

$$\therefore 250s = 81\sigma.$$

EXAMPLES.

Ex. 1. Find the number of grades in an angle of $57^\circ\ 36'\ 42''$.

$$\begin{array}{rl} 60 \,| & \underline{42{\cdot}0}\ \text{seconds} \\ 60 \,| & \underline{36{\cdot}7\ \text{minutes}} \\ & 57{\cdot}611\dot{6}\ \text{degrees} \\ \frac{1}{9} & \underline{\ 6{\cdot}401\dot{2}9\dot{6}\ } \\ & 64{\cdot}012\dot{9}6\dot{2}\ \text{grades} \\ & = 64^g\ 1^{`}\ 29{\cdot}\dot{6}2\dot{9}^{``} \end{array} \qquad (\text{lv. } 2.)$$

Ex. 2. A, B, and C are the angles of a triangle; the number of grades in A and C is four times the number of those in B, and the number of degrees in B and C is double that in A. Find the angle in degrees, grades, and circular measure.

$$\left.\begin{aligned} A+B+C&=180^\circ \\ A+C&=4B \\ B+C&=2A \end{aligned}\right\} \qquad (\text{ii. } 1.)$$

$$\therefore\ 5B=180^\circ,\ \therefore\ B=36^\circ=\frac{10}{9}\times 36^g=40^g=\frac{36\pi}{180}=\frac{\pi}{5}$$

$$3A=180^\circ,\ \therefore\ A=60^\circ=\frac{10}{9}\times 60^g=66\tfrac{2}{3}^g=\frac{60\pi}{180}=\frac{\pi}{3}$$

$$\therefore\ C=84^\circ=\frac{10}{9}\times 84^g=93\tfrac{1}{3}^g=\frac{84\pi}{180}=\frac{7\pi}{15}.$$

Ex. 3. If the number of degrees in an angle be equal to the number of grades in the complement of the same angle, prove that the circular measure of the angle is $\frac{5\pi}{19}$.

Let A = the angle in degrees,

Then $A=\frac{10}{9}(90-A)$ (*See* iv. 1.)

$\therefore\ 19\,A=900$ (xxvi. 1.)

$$A=\frac{900^\circ}{19}=\frac{900}{19}\times\frac{\pi}{180}=\frac{5\pi}{19}.$$

Ex. 4. Express in each of the three systems the angle between two adjacent sides of a regular hexagon.

Let A = the angle of a regular hexagon,

Then by Eucl. I. 32, $6A + 360^\circ = 12 \times 90^\circ$ (xv. 1.)

$$6A = 720^\circ$$

$$A = 120^\circ$$

$$= \frac{10}{9} \times 120^g = 133\tfrac{1}{3}^g$$

$$= 120 \times \frac{\pi}{180} = \frac{2\pi}{3}.$$

N.B.—If α, β, γ be the number of degrees, grades, and units of circular measure in an angle of a regular polygon of n sides,

$$\therefore\ n\alpha + 360^\circ = 2n \times 90^\circ\ \therefore\ \alpha = \frac{n-2}{n} \times 180^\circ$$

$$n\beta + 400^g = 2n \times 100^g\ \therefore\ \beta = \frac{n-2}{n} \times 200^g$$

$$n\gamma + 2\pi \quad = 2n \times \frac{\pi}{2}\ \therefore\ \gamma = \frac{n-2}{n} \times \pi.$$

Ex. 5. One angle of a triangle is π°, another is π^g, find the circular measure of the third angle.

$$\text{Circular measure of } \pi^\circ = \pi \times \frac{\pi}{180} = \frac{\pi^2}{180}$$

$$\text{Circular measure of } \pi^g = \pi \times \frac{\pi}{200} = \frac{\pi^2}{200}$$

$$\text{Sum of these two angles} = \frac{19\pi^2}{1800}.$$

$$\text{Sum of the three angles} = \pi\,;$$

$$\therefore\ \text{third angle} = \pi - \frac{19\pi^2}{1800}.$$

Ex. 6. Find the number of seconds in the angle subtended at the centre of a circle, whose radius is 1 mile, by an arc $5\tfrac{1}{2}$ inches long.

$$\text{Circular measure} = \frac{\text{arc}}{\text{rad}} = \frac{5\tfrac{1}{2}}{1760 \times 3 \times 12} = \frac{1}{11520} \qquad \text{(xxxv. 1.)}$$

$$\text{Degrees} = \frac{180}{\pi} \times \frac{1}{11520} = \frac{180}{11520} \times \frac{7}{22} = \frac{7}{1408}$$

$$\text{Seconds} = 60 \times 60 \times \frac{7}{1408} = \frac{1575}{88} = 17\tfrac{79}{88}''.$$

Ex. 7. Find the distance in miles between two places on the equator, which differ in longitude by 6° 18′, assuming the earth's equatorial diameter to be 7925·6 miles. $\left(\pi = \frac{22}{7}\right)$

$$6^\circ\ 18' = 6\tfrac{3}{10}^\circ.$$

$$\frac{\text{arc}}{\text{rad}} = \frac{63}{10} \times \frac{\pi}{180}. \qquad \text{(iv. 1.)}$$

$$\therefore \text{arc} = \frac{79256}{20} \times \frac{63}{1800} \times \frac{22}{7}$$

$$= \frac{435908}{1000} = 435{\cdot}908 \text{ miles.}$$

III.

The Trigonometrical Ratios.

1. Def. **If a point be taken in one of the straight lines which contain an angle, and a perpendicular be drawn on the other line or that line produced, the ratio of the perpendicular to the hypothenuse is called the sine of the angle**; the ratio of the base to the hypothenuse is called the *cosine;* the ratio of the perpendicular to the base is called the *tangent;* the ratio of the base to the perpendicular is called the *cotangent;* the ratio of the hypothenuse to the base is called the *secant;* the ratio of the hypothenuse to the perpendicular is called the *cosecant.*

$$\text{vers } A = 1 - \cos A, \text{ covers } A = 1 - \sin A, \text{ chord } A = 2 \sin \frac{A}{2}.$$

These trig-ratios are all *abstract numbers,* (*i.e.*) simply numerical quantities; and they remain unchanged so long as the angle remains unchanged.

Since the hypothenuse is the greatest side of a right-angled triangle, *the sin and the cos are never* >1, *the sec and cosec never* (*numerically*) <1, *the tan and cot may have any values, large or small.*

2. It is clear from the definitions that the cosec, cot, and sec are the reciprocals of the sin, tan, and cos respectively;

$$\therefore \operatorname{cosec} A = \frac{1}{\sin A}, \text{ or } \sin A = \frac{1}{\operatorname{cosec} A} \quad . \quad . \quad . \quad (1)$$

$$\sec A = \frac{1}{\cos A}, \text{ or } \cos A = \frac{1}{\sec A} \quad . \quad . \quad . \quad (2)$$

$$\tan A = \frac{1}{\cot A}, \text{ or } \cot A = \frac{1}{\tan A} \quad . \quad . \quad . \quad (3)$$

3. Also $\tan A = \frac{\sin A}{\cos A}$, $\cot A = \frac{\cos A}{\sin A}$ . . . . . (4)

$$\sec^2 A = 1 + \tan^2 A \quad (5)$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A \quad (6)$$

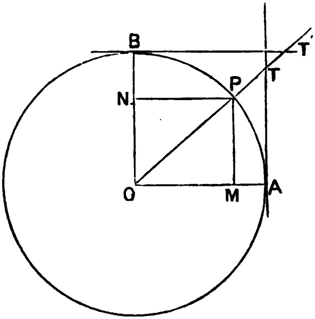
$$\left.\begin{array}{l}\sin^2 A + \cos^2 A = 1 \\ \textit{or},\ \sin A = \pm\sqrt{1-\cos^2 A},\ \cos A = \pm\sqrt{1-\sin^2 A}\end{array}\right\} \quad (7)$$

In (7), if the angle A is known, the proper sign can be given to the radical. Thus in the first and second quadrants we know that $\sin A$ is positive, $\therefore \sin A = +\sqrt{1-\cos^2 A}$; in the third and fourth quadrants $\sin A$ is negative, $\therefore \sin A = -\sqrt{1-\cos^2 A}$.

4. *Old definitions of the Trigonometrical functions of an Arc.*

The names sine, cosine, tangent, etc., were originally applied to straight *lines* connected with the *arc* subtending the angle at the centre of a circle.

If AP be the arc of a circle subtending the angle AOP at the centre, the line PM was called the sine of the arc.



AT the tangent.

OT the secant.

The *co*sine, *co*tangent, and *co*secant are the sine, tangent, and secant respectively of BP, the complement of the arc AP.

Thus PN or OM was the cosine,

BT' the cotangent,

OT' the cosecant.

The radius − the cosine $= AM$ = the versin.

The radius − the sine $= BN =$ the covers.

The line joining A, P is the chord of the arc.

The old trigonometrical functions were straight lines depending on the magnitude of the radius of the circle.

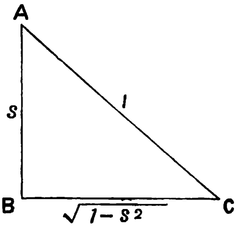
The trigonometrical ratios are abstract quantities depending only on the magnitude of the angle.

$$\text{Sine of arc } AP = PM$$

$$\text{Sine of angle } POA = \frac{PM}{OP} = \frac{PM}{\text{radius}}$$

$$\therefore \text{ sine of arc} = \text{radius} \times \text{sine of angle.}$$

5. **Prop.** Express all the trig-ratios in terms of the sine.



Let ABC be a right-angled triangle.

Denote ACB by A.

Denote AB by *s linear units* (where $s = \sin A$), and AB by 1 linear unit.

Then $BC^2 = 1 - s^2$

$$BC = \sqrt{1 - s^2}.$$

Now read off the trig-ratios from the figure.

$$\cos A = \sqrt{1 - \sin^2 A}\,;\ \tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}}\,;\ \sec A = \frac{1}{\sqrt{1 - \sin^2 A}},$$

etc. etc.

The double sign occurs before the radical, because there is more than one angle which has the given sine. If the angle A is known, the proper sign can be affixed to the radical, and there is no ambiguity.

(*See* ix. Prop. 1.)

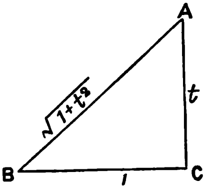
6. **Prop.** Express all the trig-ratios in terms of the tangent.

Denote AC by *t linear units* (where $t = \tan A$), and BC by 1 linear unit.

Then $AB^2 = 1 + t^2$, $AB = \sqrt{1 + t^2}$.

Now read off the trig-ratios from the figure,

$$\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}}\,;\ \cos A = \frac{1}{\sqrt{1 + \tan^2 A}}, \text{ etc.}$$



The double sign occurs before the radical, because there is more than one angle which has the given tangent. If the angle A is known, the proper sign can be affixed to the radical, and there is no ambiguity.

e.g. If A lie between 180° and 270°, we know that $\sin A$ is negative,

$$\text{and } \therefore \sin A = -\frac{\tan A}{\sqrt{1 + \tan^2 A}}. \qquad \text{(iv. 2.)}$$

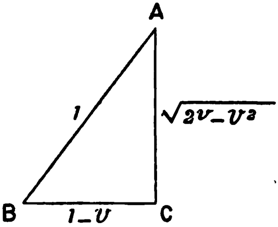
7. **Prop.** Express all the trig-ratios in terms of the vers.

vers $A=1-\cos A$, $\therefore \cos A=1-\text{vers } A$.

Denote BC by $1-v$ *linear units* (where $v=\text{vers } A$), and AB by 1 linear unit.

Then $AC^2=1-(1-v)^2=2v-v^2$

$$\therefore AC=\sqrt{2v-v^2}.$$



Now read off the trig-ratios from the figure:

$$\sin A=\sqrt{2\text{ vers } A-\text{vers}^2 A}$$

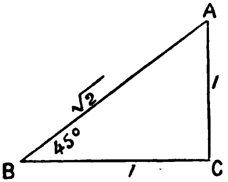
$$\tan A=\frac{\sqrt{2\text{ vers } A-\text{vers}^2 A}}{1-\text{vers } A}, \text{ etc.}$$

The double sign occurs before the radical, because there is more than one angle which has the given vers. If the angle A is known, the proper sign can be affixed to the radical, and there is no ambiguity.

8. **Prop.** Given one of the trig-ratios of an angle, to find all the others.

Having given one of the trig-ratios, we know *the relative lengths* of two of the sides of a right-angled triangle, and by Euc. I. 47 the third side can be found, and the remaining trig-ratios can be read off from the figure. (*See* Ex. 1, 3, 4.)

9. **Prop.** Find the trig-ratios of 45°.



Let ABC be a right-angled isosceles triangle, having C the right angle, and $AC=BC$: then each of the angles at A and B is 45°.

Denote BC by 1 linear unit, then $AC=1$, and

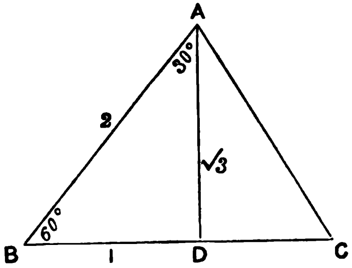
$$AB^2=1^2+1^2=2, \therefore AB=\sqrt{2}.$$

Now read off the trig-ratios from the figure;

$$\sin 45^\circ=\frac{1}{\sqrt{2}}=\cos 45^\circ, \ \tan 45^\circ=1=\cot 45^\circ, \ \sec 45^\circ=\sqrt{2}=\text{cosec } 45^\circ,$$

taking the + sign with the radical, since all the trig-ratios of 45° are +.

10. **Prop.** Find the trig-ratios of 60° and 30°.



Let *ABC* be an equilateral triangle, then each angle is 60°. Draw *AD* perpendicular to *BC*; this line bisects *BC* and the angle *BAC*, $\therefore BAD = 30°$.

Denote *BD* by 1 linear unit. Then $AB = BC = 2\,BD = 2$,

$$AD^2 = 2^2 - 1^2 = 3, \therefore AD = \sqrt{3}.$$

Now read off the trig-ratios from the figure.

$$\sin 60° = \frac{\sqrt{3}}{2} = \cos 30°,\ \cos 60° = \frac{1}{2} = \sin 30°,\ \tan 60° = \sqrt{3} = \cot 30°$$

$$\sec 60° = 2 = \operatorname{cosec} 30°,\ \operatorname{cosec} 60° = \frac{2}{\sqrt{3}} = \sec 30°,\ \cot 60° = \frac{1}{\sqrt{3}} = \tan 30°$$

taking the + sign with the radicals, since all the trig-ratios of 30° and 60° are +.

Examples.

Ex. 1. If the sine of an angle $< 90°$ is $\frac{3}{5}$, find the cosine, tangent, and secant.

Denote the perpendicular by 3, and the hypothenuse by 5,

$$\text{then } (\text{base})^2 = 5^2 - 3^2 = 16,$$

$$\therefore \text{base} = 4 \qquad \text{(xxiii. 2.)}$$

$$\therefore \cos A = \frac{4}{5},\ \tan A = \frac{3}{4},\ \sec A = \frac{5}{4}.$$

Ex. 2. Show that $\dfrac{1+\cot 60°}{1-\cot 60°} = \left\{\dfrac{1+\cos 30°}{1-\cos 30°}\right\}^{\frac{1}{2}}$

$$\text{Right-hand member} = \left\{\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right\}^{\frac{1}{2}} \qquad \text{(xli. 4.)}$$

$$= \left\{\frac{2+\sqrt{3}}{2-\sqrt{3}}\right\}^{\frac{1}{2}}$$

$$= \frac{\frac{\sqrt{3}+1}{\sqrt{2}}}{\frac{\sqrt{3}-1}{\sqrt{2}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{1+\cot 60°}{1-\cot 60°}.$$

Ex. 3. If $\sin\theta = \dfrac{a(a+2b)}{a^2+2ab+2b^2}$, find $\sec\theta$ and $\tan\theta$.

Take a right-angled triangle, and denote the perpendicular by $a(a+2b)$, and the hypothenuse by $a^2+2ab+2b^2$;

$$\therefore (\text{base})^2 = (a^2+2ab+2b^2)^2 - (a^2+2ab)^2$$

$$= (2a^2+4ab+2b^2)(2b^2) \qquad \textit{N.B.} \quad (x^2-y^2) = (x+y)(x-y)$$

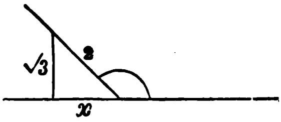
$$= 4b^2(a^2+2ab+b^2)$$

$$\text{base} = 2b(a+b) \qquad \text{(xcix. 1.)}$$

$$\therefore \sec\theta = \frac{a^2+2ab+2b^2}{2b(a+b)}, \quad \tan\theta = \frac{a(a+2b)}{2b(a+b)}.$$

Ex. 4. If the cosecant of an angle between 90° and 180° is $\dfrac{2}{\sqrt{3}}$, what is the secant? And if the cosine of an angle between 540° and 630° is $-\dfrac{1}{2}$, what is the cosecant?

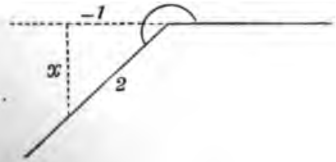
Denote the hypothenuse by 2, and the perpendicular by $\sqrt{3}$;



$$\therefore x^2 = 4 - 3 = 1 \qquad \text{(v. 1.)}$$

$$\therefore x = \pm 1, \text{ in this case} = -1$$

$$\therefore \sec a = \frac{2}{-1} = -2.$$



$$x^2 = 4 - 1 = 3$$

$$\therefore x = \pm\sqrt{3}, \text{ in this case} = -\sqrt{3}$$

$$\therefore \operatorname{cosec} a = \frac{2}{-\sqrt{3}} = -\frac{2}{\sqrt{3}}.$$

IV.

Complement, Supplement, Variations of the Trig-ratios.

1. DEF. **The Complement of an angle is its defect from a right angle; or two angles are complementary when their sum makes up a right angle.**

The complements of A, 30°, $\frac{3\pi}{4}$ are $90^\circ - A$, 60°, $-\frac{\pi}{4}$ respectively.

In a right-angled triangle ABC, in which C is the right angle, each of the acute angles is the complement of the other, $A = 90^\circ - B$, or $B = 90^\circ - A$.

2. PROP. The cosine, cotangent, and cosecant of an angle are respectively the sine, tangent, and secant of the complement.

Hence the origin of the terms *co*sine, *co*tangent, etc.

$$\cos A = \sin (90^\circ - A) \text{ or } \sin A = \cos (90^\circ - A) \quad . \quad (1)$$

$$\cot A = \tan (90^\circ - A) \text{ or } \tan A = \cot (90^\circ - A) \quad . \quad (2)$$

$$\operatorname{cosec} A = \sec (90^\circ - A) \text{ or } \sec A = \operatorname{cosec} (90^\circ - A) \quad . \quad (3)$$

If A, B, C are the angles of a triangle, $A + B + C = 180^\circ$;

$$\therefore \frac{A+B}{2} = 90^\circ - \frac{C}{2}, \therefore \sin \frac{A+B}{2} = \cos \frac{C}{2} \text{ or } \cos \frac{A+B}{2} = \sin \frac{C}{2}.$$

3. DEF. **The Supplement of an angle is its defect from two right angles, or, two angles are supplementary when their sum makes up 180°.**

The supplements of A, 30°, $\frac{3\pi}{4}$ are $180^\circ - A$, 150°, $\frac{\pi}{4}$ respectively.

If A, B, C are the angles of a triangle, $A + B + C = 180^\circ$, the sum of any two angles is the supplement of the third angle.

$$\therefore \sin (A + B) = \sin (180^\circ - C) = \sin C.$$

$$\cos (A + B) = \cos (180^\circ - C) = -\cos C.$$

4. PROP. Relations between the trig-ratios of A and $180°—A$.

$$\sin(180°-A)=\sin A,\quad \operatorname{cosec}(180°-A)=\operatorname{cosec}A \tag{4}$$
$$\cos(180°-A)=-\cos A,\quad \sec(180°-A)=-\sec A \tag{5}$$
$$\tan(180°-A)=-\tan A,\quad \cot(180°-A)=-\cot A \tag{6}$$

5. PROP. Relations between the trig-ratios of A and $180°+A$.

$$\sin(180°+A)=-\sin A,\quad \operatorname{cosec}(180°+A)=-\operatorname{cosec}A \tag{7}$$
$$\cos(180°+A)=-\cos A,\quad \sec(180°+A)=-\sec A \tag{8}$$
$$\tan(180°+A)=\tan A,\quad \cot(180°+A)=\cot A \tag{9}$$

6. PROP. Relations between the trig-ratios of A and $—A$.

$$\sin(-A)=-\sin A,\quad \operatorname{cosec}(-A)=-\operatorname{cosec}A \tag{10}$$
$$\cos(-A)=\cos A,\quad \sec(-A)=\sec A \tag{11}$$
$$\tan(-A)=-\tan A,\quad \cot(-A)=-\cot A \tag{12}$$

7. PROP. Relations between the trig-ratios of A and $90°+A$.

$$\sin(90°+A)=\cos A,\quad \operatorname{cosec}(90°+A)=\sec A \tag{13}$$
$$\cos(90°+A)=-\sin A,\quad \sec(90°+A)=-\operatorname{cosec}A \tag{14}$$
$$\tan(90°+A)=-\cot A,\quad \cot(90°+A)=-\tan A \tag{15}$$

8. PROP. Relations between the trig-ratios of A and $n\times 360°+A$, or $2n\pi+A$.

Any number of complete revolutions (four right angles) will not affect the position of the bounding line of an angle; and the trig-ratios must be the same for the same position of the revolving line. Hence *all* the trig-ratios of A and $2n\pi+A$ are the same.

$$\therefore \sin(2n\pi+A)=\sin A,\quad \cos(2n\pi+A)=\cos A, \text{ etc.}$$

9. Relations between the trig-ratios of A and $360°—A$.

$$\sin(360°-A)=\sin(-A)=-\sin A$$
$$\cos(360°-A)=\cos(-A)=\cos A, \text{ etc.}$$

10. These important results may be arranged as follows :—

$$\sin A=\sin(n\,.\,360°+A)=\sin(180°-A)=-\sin(-A)=-\sin(180°+A)$$
$$=-\sin(360°-A)=\cos(90°-A)=-\cos(90°+A)$$
$$\cos A=\cos(n\,.\,360°+A)=-\cos(180°-A)=\cos(-A)=-\cos(180°+A)$$
$$=\cos(360°-A)=\sin(90°-A)=\sin(90°+A)$$

etc., etc.

11. If A be an angle in the first quadrant—

$2n\pi + A$ represents an angle in the first quadrant;

$2n\pi + \pi - A$ or $(2n+1)\pi - A$ an angle in the second quadrant;

$2n\pi + \pi + A$ or $(2n+1)\pi + A$ an angle in the third quadrant;

$2n\pi - A$ an angle in the fourth quadrant;

where n is zero or any integer.

12. To express the trig-ratios of an angle greater than 90° in terms of the same trig-ratios of an angle in the first quadrant.

(i.) If the angle is $> 360°$, deduct any multiple of 360° from it, to make it $< 360°$.

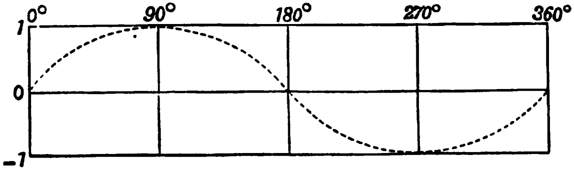
(ii.) If the angle is negative, make it positive by the formulæ connecting A and $-A$: or, add positive revolutions of four right angles (any multiple of 360°) to make it positive.

(iii.) If the angle is in the second quadrant, apply the formulæ connecting A and $180° - A$.

(iv.) If the angle is in the third quadrant, apply the formulæ connecting A and $180° + A$.

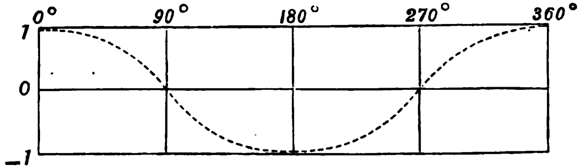
(v.) If the angle is in the fourth quadrant, apply the formulæ connecting A and $360° - A$.

13. Prop. Trace the change in sign and magnitude of the sine of an angle as the angle increases from 0° to 360°.



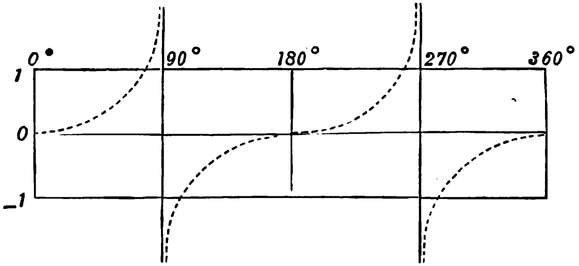
$\sin 0° = 0$, $\sin 90° = 1$, $\sin 180° = 0$, $\sin 270° = -1$, $\sin 360° = 0$.

14. **Prop.** Trace the change in sign and magnitude of the cosine of an angle as the angle increases from 0° to 360°.



cos 0° = 1, cos 90° = 0, cos 180° = − 1, cos 270° = 0, cos 360° = 1.

15. **Prop.** Trace the change in sign and magnitude of the tangent of an angle as the angle increases from 0° to 360°.



tan 0° = 0, tan 90° = + ∞ −, tan 180° = 0, tan 270° = + ∞ −, tan 360° = 0.

16. The student should also trace the changes of the cosecant, cotangent, and secant as the angle increases from 0° to 360°.

In the first quadrant all the trig-ratios are +, *in the second quadrant the sine and cosecant are* +, *the rest* − ; *in the third quadrant the sine, cos, sec, cosec, are* −, *the tan and cot* + ; *in the fourth quadrant the sine, tan, cot, and cosec are* −, *the cos and sec* +.

It should be noticed that a trig-ratio changes sign only in passing through the values 0 or ∞.

The sine and cos may have any value between − 1 and + 1.

The tan and cot may have any value between − ∞ and + ∞.

The sec and cosec may have any value between ± ∞ and ± 1.

The versin is always +, and may have any value between 0 and 2.

EXAMPLES.

Ex. 1. Find cot 240° and tan 300°. (lx. 2.)

$$\cot 240° = \cot(180° + 60°) = \cot 60° = \frac{1}{\sqrt{3}}.$$

$$\tan 300° = \tan(360° - 60°) = \tan(-60°) = -\tan 60° = -\sqrt{3}.$$

Ex. 2. Find sin 660°, tan 660°, vers 660°.

$$\sin 660° = \sin(720° - 60°) = \sin(-60°) = -\sin 60° = -\frac{\sqrt{3}}{2}.$$

$$\tan 660° = \tan(-60°) = -\tan 60° = -\sqrt{3}.$$ (lxii. 2.)

$$\text{vers } 660° = 1 - \cos 660° = 1 - \cos(-60°) = 1 - \cos 60°$$
$$= 1 - \frac{1}{2} = \frac{1}{2}.$$

Ex. 3. Prove that $\cos 30° + \cos 60° + \cos 210° + \cos 270° = \frac{1}{2}$.

Left-hand member $= \cos 30° + \cos 60° - \cos 30° + \cos 270°$

$= \cos 60°$ (lxxvii. 2.)

$= \frac{1}{2}$, since $\cos 270° = 0$.

Ex. 4. Find sec 225°, cot 435°, cosec 660°, sec 555°.

$$\sec 225° = \sec(180° + 45°) = -\sec 45° = -\sqrt{2}.$$ (li. 4.

$$\cot 435° = \cot(360° + 75°) = \cot 75° = 2 - \sqrt{3}.$$

$$\text{cosec } 660° = \text{cosec}(720° - 60°) = \text{cosec}(-60°) = -\text{cosec } 60° = -\frac{2}{\sqrt{3}}.$$

$$\sec 555° = \sec(720° - 165°) = \sec(-165°) = \sec 165°$$
$$= \sec(180° - 15°) = -\sec 15°$$ (*see* Chap. v.).

$$= -\frac{2\sqrt{2}}{\sqrt{3}+1} = -\frac{2\sqrt{2}(\sqrt{3}-1)}{2} = -\sqrt{2}(\sqrt{3}-1).$$

V.

Trig-ratios of Compound Angles.

N.B.—Sin $(A+B)$ is the sine of the compound angle $(A+B)$, and is not equal to $\sin A + \sin B$: *e.g.* if $A=30°$, $B=60°$,

$$\sin(A+B)=\sin 90°=1, \quad \sin A+\sin B=\sin 30°+\sin 60°=\frac{1}{2}+\frac{\sqrt{3}}{2}.$$

$$\sin(A \pm B)=\sin A\cos B \pm \cos A\sin B \quad . \quad . \quad . \quad . \quad (1)$$

$$\sin 15°=\frac{\sqrt{3}-1}{2\sqrt{2}}=\cos 75° \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\sin 75°=\frac{\sqrt{3}+1}{2\sqrt{2}}=\cos 15° \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\cos(A \pm B)=\cos A\cos B \mp \sin A\sin B \quad . \quad . \quad . \quad . \quad (4)$$

$$\tan(A \pm B)=\frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\tan 15°=2-\sqrt{3}=\cot 75° \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\tan 75°=2+\sqrt{3}=\cot 15° \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\tan(45° \pm A)=\frac{1 \pm \tan A}{1 \mp \tan A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\cot(A \pm B)=\frac{\cot A\cot B \mp 1}{\cot B \pm \cot A} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B \quad . \quad . \quad . \quad . \quad (10)$$

$$\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A \quad . \quad (11)$$

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \quad . \quad . \quad . \quad . \quad (12)$$

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \quad . \quad . \quad . \quad . \quad (13)$$

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \quad . \quad . \quad . \quad . \quad (14)$$

$$\cos B-\cos A=2\sin\frac{A+B}{2}\sin\frac{A-B}{2} \quad . \quad . \quad . \quad . \quad (15)$$

or, sum of two sines = twice the sine of semi-sum × cos of semi-difference,
difference of two sines = twice the cos of semi-sum × sine of semi-difference,
sum of two cosines = twice the cos of semi-sum × cos of semi-difference,
difference of two cosines = twice the sine of semi-sum × sine of semi-difference.

The converse of these formulæ should be carefully learnt :—

$$\sin \times \cos = \frac{1}{2}\left\{\sin(\text{sum}) + \sin(\text{difference})\right\}$$

$$\cos \times \sin = \frac{1}{2}\left\{\sin(\text{sum}) - \sin(\text{difference})\right\}$$

$$\cos \times \cos = \frac{1}{2}\left\{\cos(\text{sum}) + \cos(\text{difference})\right\}$$

$$\sin \times \sin = \frac{1}{2}\left\{\cos(\text{difference}) - \cos(\text{sum})\right\}.$$

2. The formulæ in Ch. IV. may all be proved by means of the $A \pm B$ formulæ, *e.g.*

$$\sin(90^\circ - A) = \sin 90^\circ \cos A - \cos 90^\circ \sin A = \cos A.$$

$$\cos(90^\circ + A) = \cos 90^\circ \cos A - \sin 90^\circ \sin A = -\sin A.$$

$$\tan(180^\circ + A) = \frac{\tan 180^\circ + \tan A}{1 - \tan 180^\circ \tan A} = \tan A.$$

Since $\sin 90^\circ = 1$, $\cos 90^\circ = 0$, $\tan 180^\circ = 0$.

3. Having proved $\sin(A+B) = \sin A \cos B + \cos A \sin B$, deduce $\sin(A-B)$, $\cos(A+B)$, $\cos(A-B)$.

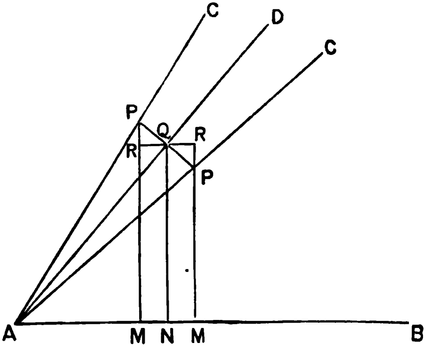
$$\sin(A-B) = \sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$
$$= \sin A \cos B - \cos A \sin B.$$

$$\cos(A+B) = \sin(90^\circ + \overline{A+B}) = \sin(\overline{90^\circ + A} + B) \qquad \text{(cvi. 1.)}$$
$$= \sin(90^\circ + A)\cos B + \cos(90^\circ + A)\sin B$$
$$= \cos A \cos B - \sin A \sin B.$$

$$\cos(A-B) = \sin(90^\circ - \overline{A-B}) = \sin(\overline{90^\circ - A} + B)$$
$$= \sin(90^\circ - A)\cos B + \cos(90^\circ - A)\sin B$$
$$= \cos A \cos B + \sin A \sin B.$$

By a similar process the student should *deduce* the other three formulæ from cos $(A+B)$ or sin $(A-B)$.

4. **Prop.** Express the sine, cosine, and tangent of a compound angle in terms of the sines, cosines, and tangents of the single angles.



Let $BAD=A$, $CAD=B$, $\therefore BAC=A\pm B$.

Draw PQP perpendicular to AD, and construct as in the figure.

$$QPR=90^\circ-PQR=RQA=QAB=A.$$

$$\sin(A\pm B)=\frac{PM}{AP}=\frac{QN\pm PR}{AP}=\frac{QN}{AQ}\cdot\frac{AQ}{AP}\pm\frac{PR}{PQ}\cdot\frac{PQ}{AP}$$
$$=\sin A\cos B\pm\cos A\sin B.$$

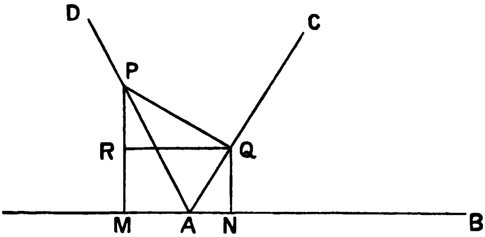
$$\cos(A\pm B)=\frac{AM}{AP}=\frac{AN\mp RQ}{AP}=\frac{AN}{AQ}\cdot\frac{AQ}{AP}\mp\frac{RQ}{PQ}\cdot\frac{PQ}{AP}$$
$$=\cos A\cos B\mp\sin A\sin B.$$

$$\tan(A\pm B)=\frac{PM}{AM}=\frac{QN\pm PR}{AN\mp RQ}=\frac{\dfrac{QN}{AN}\pm\dfrac{PR}{AN}}{1\mp\dfrac{RQ}{PR}\cdot\dfrac{PR}{AN}}$$
$$=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}.$$

Since PQR, QAN are similar triangles, $\therefore \frac{PR}{AN}=\frac{PQ}{AQ}=\tan B.$

5. **Prop.** Prove the formulæ for $\sin(A+B)$, $\cos(A+B)$, $\tan(A+B)$,

(i.) When $A<90°$, $B<90°$, and $A+B>90°$.



$$BAC=A,\quad CAD=B,\quad \therefore BAD=A+B.$$

$$RPQ=90°-PQR=RQA=QAB=A.$$

$$\sin(A+B)=\frac{PM}{AP}=\frac{QN+PR}{AP}=\frac{QN}{AQ}\cdot\frac{AQ}{AP}+\frac{PR}{PQ}\cdot\frac{PQ}{AP}$$

$$=\sin A\cos B+\cos A\sin B.$$

$$\cos(A+B)=\frac{-AM}{AP}=\frac{-(MN-AN)}{AP}=\frac{AN-RQ}{AP}$$

$$=\frac{AN}{AQ}\cdot\frac{AQ}{AP}-\frac{RQ}{PQ}\cdot\frac{PQ}{AP}$$

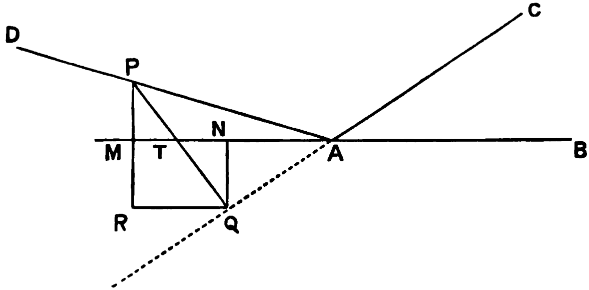
$$=\cos A\cos B-\sin A\sin B.$$

$$\tan(A+B)=\frac{PM}{-AM}=\frac{QN+PR}{-(MN-AN)}=\frac{QN+PR}{AN-RQ}$$

$$=\frac{\dfrac{QN}{AN}+\dfrac{PR}{AN}}{1-\dfrac{RQ}{PR}\cdot\dfrac{PR}{AN}}=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

Since PQR, QAN are similar triangles, $\therefore \dfrac{PR}{AN}=\dfrac{PQ}{AQ}=\tan B.$

(ii.) $A<90^\circ$, $B>90^\circ$, and $A+B<180^\circ$.



$$BAC=A,\quad CAD=B,\quad \therefore BAD=A+B.$$

$$RPQ=90^\circ-PTM=TAQ=BAC=A.$$

$$\sin(A+B)=\frac{PM}{AP}=\frac{PR-QN}{AP}=\frac{-QN+PR}{AP}=\frac{-QN}{AQ}\cdot\frac{AQ}{AP}+\frac{PR}{PQ}\cdot\frac{PQ}{AP}$$

$$=-\sin A\cos(180^\circ-B)+\cos A\sin(180^\circ-B)$$

$$=\sin A\cos B+\cos A\sin B.$$

$$\cos(A+B)=\frac{-AM}{AP}=-\frac{AN+RQ}{AP}=-\frac{AN}{AQ}\cdot\frac{AQ}{AP}-\frac{RQ}{PQ}\cdot\frac{PQ}{AP}$$

$$=-\cos A\cos(180^\circ-B)-\sin A\sin(180^\circ-B)$$

$$=\cos A\cos B-\sin A\sin B.$$

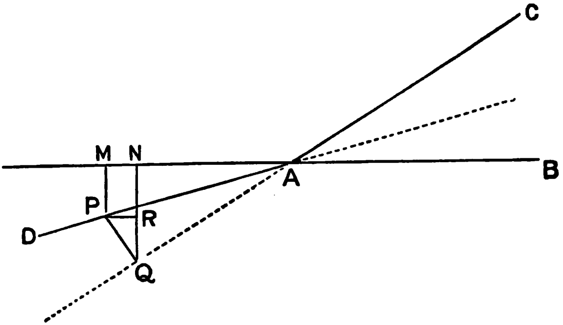
$$\tan(A+B)=\frac{PM}{-AM}=\frac{PR-QN}{-(AN+NM)}=\frac{QN-PR}{AN+RQ}=\frac{\frac{QN}{AN}-\frac{PR}{AN}}{1+\frac{RQ}{PR}\cdot\frac{PR}{AN}}$$

$$=\frac{\tan A-\tan(180^\circ-B)}{1+\tan A\tan(180^\circ-B)}$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

Since PQR, QAN are similar triangles, $\therefore \frac{PR}{AN}=\frac{PQ}{AQ}=\tan(180^\circ-B)$.

(iii.) $A < 90$, and $A + B > 180°$.



$$BAC = A,\quad CAD = B,\quad \therefore BAD = A + B.$$

$$PQR = 90° - NQA = QAN = BAC = A.$$

$$\sin(A+B) = \frac{-PM}{AP} = \frac{-(QN - QR)}{AP} = \frac{-QN}{AQ}\cdot\frac{AQ}{AP} + \frac{QR}{PQ}\cdot\frac{PQ}{AP}$$

$$= -\sin A \cos(180° - B) + \cos A \sin(180° - B)$$

$$= \sin A \cos B + \cos A \sin B.$$

$$\cos(A+B) = \frac{-AM}{AP} = \frac{-(AN + PR)}{AP} = \frac{-AN}{AQ}\cdot\frac{AQ}{AP} - \frac{PR}{PQ}\cdot\frac{PQ}{AP}$$

$$= -\cos A \cos(180° - B) - \sin A \sin(180° - B)$$

$$= \cos A \cos B - \sin A \sin B.$$

$$\tan(A+B) = \frac{PM}{AM} = \frac{QN - QR}{AN + PR} = \frac{\dfrac{QN}{AN} - \dfrac{QR}{AN}}{1 + \dfrac{PR}{QR}\cdot\dfrac{QR}{AN}}$$

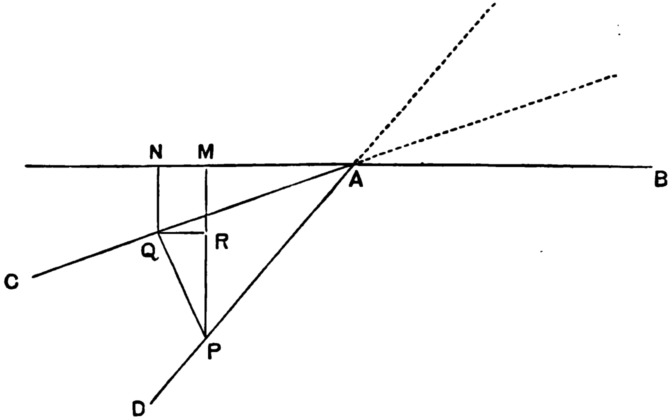
$$= \frac{\tan A - \tan(180° - B)}{1 + \tan A \tan(180° - B)} = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Since PQR, QAN are similar triangles, $\therefore \dfrac{QR}{AN} = \dfrac{PQ}{AQ} = \tan(180° - B)$.

(iv.) $A>180^\circ$, $A+B>180^\circ<270^\circ$.

$$BAC=A,\ CAD=B,\ \therefore BAD=A+B.$$

$$QPR=90^\circ-PQR=RQA=QAN=A-180^\circ.$$



$$\sin(A+B)=\frac{-PM}{AP}=\frac{-(QN+PR)}{AP}=\frac{-QN}{AQ}\cdot\frac{AQ}{AP}-\frac{PR}{PQ}\cdot\frac{PQ}{AP}$$

$$=-\sin(A-180^\circ)\cos B-\cos(A-180^\circ)\sin B$$

$$=\ \sin A\cos B+\cos A\sin B.$$

$$\cos(A+B)=\frac{-AM}{AP}=\frac{-(AN-QR)}{AP}=\frac{-AN}{AQ}\cdot\frac{AQ}{AP}+\frac{QR}{PQ}\cdot\frac{PQ}{AP}$$

$$=-\cos(A-180^\circ)\cos B+\sin(A-180^\circ)\sin B$$

$$=\ \cos A\cos B-\sin A\sin B.$$

$$\tan(A+B)=\frac{-PM}{-AM}=\frac{QN+PR}{AN-RQ}=\frac{\frac{QN}{AN}+\frac{PR}{AN}}{1-\frac{RQ}{PR}\cdot\frac{PR}{AN}}$$

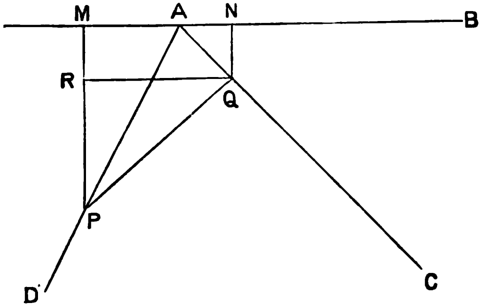
$$=\frac{\tan(A-180^\circ)+\tan B}{1-\tan(A-180^\circ)\tan B}=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

Since PQR, QAN are similar triangles, $\therefore \frac{PR}{AN}=\frac{PQ}{AQ}=\tan B$.

6. Prop. Prove the formulæ for sin $(A-B)$, cos $(A-B)$, tan $(A-B)$ when A is in the fourth quadrant, and $A-B$ in the third.

$$BAC=A,\quad DAC=B,\quad \therefore BAD=A-B.$$

$$QPR=90^\circ-PQR=RQA=QAN=360^\circ-A.$$



$$\sin(A-B)=-\frac{PM}{AP}=-\frac{QN+PR}{AP}=-\frac{QN}{AQ}\cdot\frac{AQ}{AP}-\frac{PR}{PQ}\cdot\frac{PQ}{AP}$$

$$=-\sin(360^\circ-A)\cos B-\cos(360^\circ-A)\sin B$$

$$=\sin A\cos B-\cos A\sin B.$$

$$\cos(A-B)=-\frac{AM}{AP}=-\frac{RQ-AN}{AP}=\frac{AN-RQ}{AP}=\frac{AN}{AQ}\cdot\frac{AQ}{AP}-\frac{RQ}{PQ}\cdot\frac{PQ}{AP}$$

$$=\cos(360^\circ-A)\cos B-\sin(360^\circ-A)\sin B$$

$$=\cos A\cos B+\sin A\sin B.$$

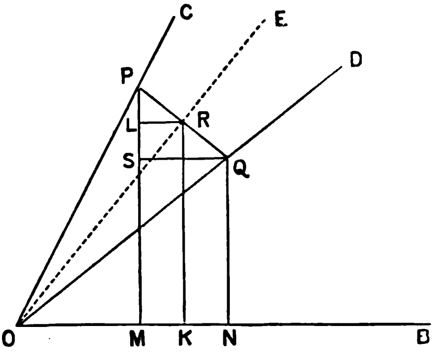
$$\tan(A-B)=\frac{-PM}{-AM}=\frac{QN+PR}{RQ-AN}=\frac{QN+PR}{-AN+RQ}=\frac{\dfrac{QN}{AN}+\dfrac{PR}{AN}}{-1+\dfrac{RQ}{PR}\cdot\dfrac{PR}{AN}}$$

$$=\frac{\tan(360^\circ-A)+\tan B}{-1+\tan(360^\circ-A)\tan B}$$

$$=\frac{-\tan A+\tan B}{-1-\tan A\tan B}=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

Since PQR, QAN are similar triangles, $\therefore \dfrac{PR}{AN}=\dfrac{PQ}{AQ}=\tan B.$

7. **Prop.** Prove geometrically the formulæ for the sum or difference of two sines and cosines.



Let $BOC=A$ and $BOD=B$,

Then $COD=A-B$.

Bisect COD by OE,

Then $COE=DOE$

$$=\frac{A-B}{2}.$$

$$BOE=BOD+DOE$$

$$=B+\frac{A-B}{2}=\frac{A+B}{2}.$$

Draw PRQ perpendicular to OE, and the other perpendiculars as in the figure.

Then $OP=OQ$, $LPR=90^\circ-PRL=LRO=ROB=\frac{A+B}{2}$.

(1) $$\sin A+\sin B=\frac{PM}{OP}+\frac{QN}{OQ}=\frac{PM+QN}{OP}=\frac{2RK}{OP}=2\frac{RK}{OR}\cdot\frac{OR}{OP}$$
$$=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

(2) $$\sin A-\sin B=\frac{PM}{OP}-\frac{QN}{OQ}=\frac{PM-QN}{OP}=\frac{2PL}{OP}=2\frac{PL}{PR}\cdot\frac{PR}{OP}$$
$$=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.$$

(3) $$\cos A+\cos B=\frac{OM}{OP}+\frac{ON}{OQ}=\frac{OM+ON}{OP}=\frac{2OK}{OP}=2\frac{OK}{OR}\cdot\frac{OR}{OP}$$
$$=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

(4) $$\cos B-\cos A=\frac{ON}{OQ}-\frac{OM}{OP}=\frac{ON-OM}{OP}=\frac{2MK}{OP}=2\frac{LR}{PR}\cdot\frac{PR}{OP}$$
$$=2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

EXAMPLES.

Ex. 1. The cosines of two angles of a triangle are $\frac{3}{5}$ and $\frac{12}{13}$ respectively, find all the trig-ratios of the third angle.

$$\cos A=\frac{3}{5}, \quad \therefore \sin A=\frac{4}{5} \qquad \text{(xxxv. 3.)}$$

$$\cos B=\frac{12}{13}, \quad \therefore \sin B=\frac{5}{13}$$

$$\sin C=\sin(A+B)=\sin A\cos B+\cos A\sin B=\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}=\frac{63}{65}.$$

$$\cos C=-\cos(A+B)=\sin A\sin B-\cos A\cos B=\frac{4}{5}\cdot\frac{5}{13}-\frac{3}{5}\cdot\frac{12}{13}=-\frac{16}{65}.$$

$$\therefore \tan C=-\frac{63}{16}, \text{ etc.}$$

Ex. 2. Represent $\sin A-\cos B$ in the form of a product.

$$\begin{aligned}\sin A-\cos B&=\sin A-\sin(90^\circ-B)\\&=2\cos\frac{1}{2}(A+90^\circ-B)\sin\frac{1}{2}(A-90^\circ+B)\\&=2\cos\left(45^\circ+\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}-45^\circ\right).\end{aligned}$$

Ex. 3. Prove that $\cos 20^\circ+\cos 100^\circ+\cos 140^\circ=0$.

$$\begin{aligned}\text{Left-hand member}&=\cos 20^\circ+2\cos 120^\circ\cos 20^\circ \qquad \text{(xxxiii. 5.)}\\&=\cos 20^\circ\left\{1+2\left(-\frac{1}{2}\right)\right\}\\&=0.\end{aligned}$$

Ex. 4. Express $2\sin 5A\sin 9A$ as the difference of two cosines.

$$\begin{aligned}2\sin 5A\sin 9A&=\cos(9A-5A)-\cos(9A+5A)\\&=\cos 4A-\cos 14A. \qquad \text{(xl. 3.)}\end{aligned}$$

Ex. 5. Express $\frac{1}{2}\{\cos(3A+2B)+\cos(5A-4B)\}$ as the product of two cosines.

$$\begin{aligned}\text{expression}&=\frac{1}{2}\left\{2\cos\frac{3A+2B+5A-4B}{2}\cos\frac{3A+2B-5A+4B}{2}\right\}\\&=\cos(4A-B)\cos(3B-A). \qquad \text{(xxxix. 3.)}\end{aligned}$$

Ex. 6. Prove that—

$$\frac{\sin 3A+2\sin 5A+\sin 7A}{\sin 5A+2\sin 7A+\sin 9A}=\frac{\sin 5A}{\sin 7A}.$$

$$\text{Left-hand member}=\frac{2\sin 5A+2\sin 5A\cos 2A}{2\sin 7A+2\sin 7A\cos 2A} \qquad \text{(lxxvii. 2.)}$$

$$=\frac{2\sin 5A(1+\cos 2A)}{2\sin 7A(1+\cos 2A)}=\frac{\sin 5A}{\sin 7A}.$$

Ex. 7. Prove that—

$$\frac{\sin A+\sin(n-1)A+\sin(2n-3)A}{\cos A+\cos(n-1)A+\cos(2n-3)A}=\tan(n-1)A.$$

$$\text{Left-hand member}=\frac{\sin(n-1)A+2\sin(n-1)A\cos(n-2)A}{\cos(n-1)A+2\cos(n-1)A\cos(n-2)A}$$

$$=\frac{\sin(n-1)A\,\{2\cos(n-2)A+1\}}{\cos(n-1)A\,\{2\cos(n-2)A+1\}} \qquad \text{(xcix. 2.)}$$

$$=\tan(n-1)A.$$

Ex. 8. Trace the change in sign and magnitude of $\sin A+\cos A$, as A varies from 0° to 360°.

$$\sin A+\cos A=\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin A+\frac{1}{\sqrt{2}}\cos A\right)$$

$$=\sqrt{2}(\sin 45^\circ\cos A+\cos 45^\circ\sin A)$$

$$=\sqrt{2}\sin(45^\circ+A).$$

The expression is +, when $45^\circ+A$ lies between 0° and 180°,

i.e. when A lies between -45° and 135°.

The expression is −, when $45^\circ+A$ lies between 180° and 360°,

i.e. when A lies between 135° and 315°.

$A=-45^\circ$, $\sqrt{2}\sin(45^\circ+A)=0$,

$A=0^\circ$, „ „ „ $=1$,

$A=45^\circ$, „ „ „ $=\sqrt{2}$,

$A=90^\circ$, „ „ „ $=1$.

In the first quadrant, the value changes from 1 to $\sqrt{2}$, and then from $\sqrt{2}$ to 1.

$$A=135^\circ,\quad \sqrt{2}\sin(45^\circ+A)=0,$$
$$A=180^\circ,\quad \text{,,}\quad \text{,,}\quad \text{,,}\quad =-1.$$

In the second quadrant, the value changes from 1 to 0, and then from 0 to -1.

$$A=225^\circ,\quad \sqrt{2}\sin(45^\circ+A)=-\sqrt{2},$$
$$A=270^\circ,\quad \text{,,}\quad \text{,,}\quad \text{,,}\quad =-1.$$

In the third quadrant, the value changes from -1 to $-\sqrt{2}$, and then from $-\sqrt{2}$ to -1.

$$A=315^\circ,\quad \sqrt{2}\sin(45^\circ+A)=0,$$
$$A=360^\circ,\quad \text{,,}\quad \text{,,}\quad \text{,,}\quad =1.$$

In the fourth quadrant, the value changes from -1 to 0, and then from 0 to 1.

Ex. 9. Trace the variations in sign and magnitude of $\sin A-\cos A$, as A varies from 0° to 180°.

$$\sin A-\cos A=\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin A-\frac{1}{\sqrt{2}}\cos A\right)$$
$$=\sqrt{2}\,(\sin A\ \cos 45^\circ-\cos A\ \sin 45^\circ)$$
$$=\sqrt{2}\sin(A-45^\circ).$$

The expression is +, when $A-45^\circ$ lies between 0° and 180°,
i.e. when A lies between 45° and 225°.

The expression is $-$, when $A-45^\circ$ lies between 180° and 360°,
i.e. when A lies between 225° and 405°, or 45°.

$$A=0^\circ,\quad \sqrt{2}\sin(A-45^\circ)=-1,$$
$$A=45^\circ,\quad \text{,,}\quad \text{,,}\quad =\ 0,\qquad \text{(x. 1.)}$$
$$A=90^\circ,\quad \text{,,}\quad \text{,,}\quad =\ 1.$$

In the first quadrant, the value changes from -1 to 0, and then from 0 to 1.

$$A=135^\circ,\quad \sqrt{2}\sin(A-45^\circ)=\sqrt{2},$$
$$A=180^\circ,\quad \text{,,}\quad \text{,,}\quad =\ 1.$$

In the second quadrant, the value changes from 1 to $\sqrt{2}$, and then from $\sqrt{2}$ to 1.

Ex. 10. Trace the changes in sign and magnitude of $\sin A - \sqrt{3}\cos A$ as A increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

$$\sin A - \sqrt{3}\cos A = 2\left\{\frac{1}{2}\sin A - \frac{\sqrt{3}}{2}\cos A\right\}$$

$$= 2\left(\sin A\cos\frac{\pi}{3} - \cos A\sin\frac{\pi}{3}\right).$$

$$= 2\sin\left(A - \frac{\pi}{3}\right).$$

The expression is +, when $A - \frac{\pi}{3}$ lies between 0 and π,

i.e. when A lies between $\frac{\pi}{3}$ and $\frac{4\pi}{3}$.

The expression is −, when $A - \frac{\pi}{3}$ lies between π and 2π,

i.e. when A lies between $\frac{4\pi}{3}$ and $\frac{7\pi}{3}$, or $\frac{\pi}{3}$.

$$A = \frac{\pi}{2},\quad 2\sin\left(A - \frac{\pi}{3}\right) = 2\sin\frac{\pi}{6} = 1,$$

$$A = \frac{2\pi}{3},\quad ,, \qquad ,, \quad = 2\sin\frac{\pi}{3} = \sqrt{3}, \qquad \text{(liii. 3.)}$$

$$A = \frac{5\pi}{6},\quad ,, \qquad ,, \quad = 2\sin\frac{\pi}{2} = 2,$$

$$A = \pi,\quad ,, \qquad ,, \quad = 2\sin\frac{2\pi}{3} = \sqrt{3}.$$

In the second quadrant, the value changes from 1 to $\sqrt{3}$, and then from $\sqrt{3}$ to 2, and then decreases from 2 to $\sqrt{3}$.

$$A = \frac{7\pi}{6},\quad 2\sin\left(A - \frac{\pi}{3}\right) = 2\sin\frac{5\pi}{6} = 1,$$

$$A = \frac{4\pi}{3},\quad ,, \qquad ,, \quad = 2\sin\pi = 0,$$

$$A = \frac{3\pi}{2},\quad ,, \qquad ,, \quad = 2\sin\frac{7\pi}{6} = -1.$$

In the third quadrant, the value changes from $\sqrt{3}$ to 1 and then to −1, passing through the value 0, when $A = \frac{4\pi}{3}$.

VI.

Trig-ratios of Multiple Angles.

1. $\operatorname{Sin} 2A = 2\sin A\cos A,\quad \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$ . . (1)

$$\left.\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A\\ \cos A &= \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}\end{aligned}\right\}\quad . \quad (2)$$

$$2\cos^2 A = 1 + \cos 2A,\quad 2\cos^2\frac{A}{2} = 1 + \cos A \quad . \quad . \quad . \quad (3)$$

$$2\sin^2 A = 1 - \cos 2A,\quad 2\sin^2\frac{A}{2} = 1 - \cos A \quad . \quad . \quad . \quad (4)$$

$$\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A},\quad \tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A} \quad . \quad . \quad . \quad . \quad (5)$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A},\quad \tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} \quad . \quad . \quad . \quad . \quad (6)$$

$$\cot 2A = \frac{\cot^2 A - 1}{2\cot A},\quad \cot A = \frac{\cot^2\frac{A}{2} - 1}{2\cot\frac{A}{2}} \quad . \quad . \quad . \quad . \quad (7)$$

$$\tan A = \frac{\sin 2A}{1+\cos 2A},\quad \tan\frac{A}{2} = \frac{\sin A}{1+\cos A} \quad . \quad . \quad . \quad . \quad (8)$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A},\quad \sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} \quad . \quad . \quad . \quad . \quad (9)$$

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A},\quad \cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}} \quad . \quad . \quad . \quad . \quad (10)$$

$$\operatorname{cosec} 2A - \cot 2A = \tan A,\quad \operatorname{cosec} A - \cot A = \tan\frac{A}{2} \quad . \quad . \quad (11)$$

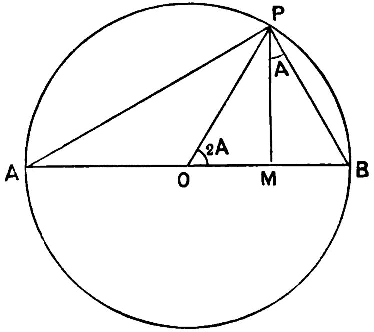
$$\operatorname{cosec} 2A + \cot 2A = \cot A,\quad \operatorname{cosec} A + \cot A = \cot\frac{A}{2} \quad . \quad . \quad (12)$$

$$\sin 3A = 3\sin A - 4\sin^3 A \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\cos 3A = 4\cos^3 A - 3\cos A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

2. Prop. Trig-ratios of $2A$ in terms of those of A.



The angle POB at the centre is double the angle PAB at the circumference.

If $PAM=A$, $POM=2A$.

And since APB in a semicircle is a right angle,

$$\therefore BPM=90^\circ-APM$$
$$=A.$$

Also
$$2OM=2(AM-AO)$$
$$=2AM-AB$$
$$=2AM-(AM+MB)$$
$$=AM-MB.$$

(1) $$\sin 2A=\frac{PM}{OP}=\frac{2PM}{2OP}=\frac{2PM}{AB}=\frac{2PM}{AP}\cdot\frac{AP}{AB}=2\sin A\cos A.$$

(2) $$\cos 2A=\frac{OM}{OP}=\frac{2OM}{2OP}=\frac{AM-MB}{AB}=\frac{AM}{AP}\cdot\frac{AP}{AB}-\frac{MB}{BP}\cdot\frac{BP}{AB}$$

$$=\cos A\cos A-\sin A\sin A=\cos^2 A-\sin^2 A\;;$$

$$\cos 2A=\frac{2OM}{2OP}=\frac{2AM-AB}{AB}=\frac{2AM}{AP}\cdot\frac{AP}{AB}-1=2\cos^2 A-1\;;$$

$$\cos 2A=\frac{2OM}{2OP}=\frac{2(OB-MB)}{AB}=\frac{AB-2MB}{AB}=1-\frac{2MB}{BP}\cdot\frac{BP}{AB}$$

$$=1-2\sin^2 A.$$

(3) $$\tan 2A=\frac{2PM}{2OM}=\frac{2PM}{AM-MB}=\frac{\dfrac{2PM}{AM}}{1-\dfrac{MB}{PM}\cdot\dfrac{PM}{AM}}=\frac{2\tan A}{1-\tan^2 A}.$$

(4) $$\frac{\sin 2A}{1+\cos 2A}=\frac{\dfrac{PM}{OP}}{1+\dfrac{OM}{OP}}=\frac{PM}{OP+OM}=\frac{PM}{AO+OM}=\frac{PM}{AM}=\tan A.$$

(5) $$\frac{1-\cos 2A}{1+\cos 2A}=\frac{1-\dfrac{OM}{OP}}{1+\dfrac{OM}{OP}}=\frac{OP-OM}{OP+OM}=\frac{OB-OM}{AO+OM}=\frac{MB}{AM}$$

$$=\frac{MB}{PM}\cdot\frac{PM}{AM}=\tan^2 A.$$

3. **Prop.** Given $\sin A$, find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$.

$$\left.\begin{aligned}\text{Sin } A&=2\sin\frac{A}{2}\cos\frac{A}{2}\\ 1&=\sin^2\frac{A}{2}+\cos^2\frac{A}{2}\end{aligned}\right\}$$

$$\left.\begin{aligned}1+\sin A&=\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2\\ 1-\sin A&=\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2\end{aligned}\right\}$$

$$\left.\begin{aligned}\sin\frac{A}{2}+\cos\frac{A}{2}&=\pm\sqrt{1+\sin A}\\ \sin\frac{A}{2}-\cos\frac{A}{2}&=\pm\sqrt{1-\sin A}\end{aligned}\right\}$$

$$\left.\begin{aligned}\therefore 2\sin\frac{A}{2}&=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}\\ 2\cos\frac{A}{2}&=\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}\end{aligned}\right\}$$

Four values, according as we take the various combinations of the signs before each of the radicals.

If $\sin A$ be given, and nothing more is known respecting A, the ambiguities of sign cannot be removed. But if the angle A is given, or if we know the quadrant in which $\frac{A}{2}$ lies, we can determine the proper signs of the radicals.

In all examples write down the original equations

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A} \qquad \text{(i.)}$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A} \qquad \text{(ii.)}$$

Then as we know A, we can find the signs of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ and which is the greater. Having found the sign of the left-hand member, we affix the same sign to the radical on the right side.

From $-\frac{\pi}{4}$ to $\frac{\pi}{4}$, the cos > sine, and is +, ∴ the sign of (i) is +,

and the sign of (ii) is −.

From $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, the sin > cos, and is +, ∴ the sign of (i) is +,

and the sign of (ii) is +. (xciii. 1.)

From $\frac{3\pi}{4}$ to $\frac{5\pi}{4}$, the cos > sin, and is −, ∴ the sign of (i) is −,

and the sign of (ii) is +.

From $\frac{5\pi}{4}$ to $\frac{7\pi}{4}$, the sin > cos, and is −, ∴ both signs are −.

EXAMPLES.

Ex. 1. Find the trig-ratios of $22\frac{1}{2}°$.

$$2\sin^2 22\tfrac{1}{2}° = 1-\cos 45° = 1-\frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}} = \frac{2-\sqrt{2}}{2}$$

$$\sin^2 22\tfrac{1}{2}° = \frac{2-\sqrt{2}}{4}$$

$$\therefore\ \sin 22\tfrac{1}{2}° = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

$$2\cos^2 22\tfrac{1}{2}° = 1+\cos 45° = 1+\frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}} = \frac{2+\sqrt{2}}{2}$$

$$\cos^2 22\tfrac{1}{2}° = \frac{2+\sqrt{2}}{4}$$

$$\therefore\ \cos 22\tfrac{1}{2}° = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

$$\tan 22\tfrac{1}{2}° = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} = \sqrt{(\sqrt{2}-1)^2} = \sqrt{2}-1.$$

Ex. 2. Prove that $\tan 7\tfrac{1}{2}^\circ=(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$.

$$\tan^2 7\tfrac{1}{2}^\circ=\frac{1-\cos 15^\circ}{1+\cos 15^\circ}=\frac{1-\dfrac{\sqrt{3}+1}{2\sqrt{2}}}{1+\dfrac{\sqrt{3}+1}{2\sqrt{2}}}=\frac{2\sqrt{2}-\sqrt{3}-1}{2\sqrt{2}+\sqrt{3}+1}$$

$$=\frac{(\overline{2\sqrt{2}-\sqrt{3}}-1)(\overline{2\sqrt{2}-\sqrt{3}}+1)}{(\overline{2\sqrt{2}+1}+\sqrt{3})(\overline{2\sqrt{2}+1}-\sqrt{3})} \qquad \text{(xci. 1.}$$

$$=\frac{10-4\sqrt{6}}{(9+4\sqrt{2})-3}=\frac{5-2\sqrt{6}}{3+2\sqrt{2}}=\frac{(5-2\sqrt{6})(3-2\sqrt{2})}{9-8}$$

$$=(5-2\sqrt{6})(3-2\sqrt{2})$$

$$\therefore\ \tan 7\tfrac{1}{2}^\circ=(\sqrt{3}-\sqrt{2})(\sqrt{2}-1).$$

Ex. 3. Prove that $\tan 37^\circ\,30'=\sqrt{6}+\sqrt{3}-\sqrt{2}-2$.

$$\tan^2 37^\circ\,30'=\frac{1-\cos 75^\circ}{1+\cos 75^\circ}=\frac{1-\dfrac{\sqrt{3}-1}{2\sqrt{2}}}{1+\dfrac{\sqrt{3}-1}{2\sqrt{2}}}=\frac{2\sqrt{2}-\sqrt{3}+1}{2\sqrt{2}+\sqrt{3}-1}$$

$$=\frac{(\overline{2\sqrt{2}-\sqrt{3}}+1)(\overline{2\sqrt{2}-\sqrt{3}}-1)}{(\overline{2\sqrt{2}-1}+\sqrt{3})(\overline{2\sqrt{2}-1}-\sqrt{3})}=\frac{(2\sqrt{2}-\sqrt{3})^2-1}{(2\sqrt{2}-1)^2-3}$$

$$=\frac{10-4\sqrt{6}}{6-4\sqrt{2}}=\frac{5-2\sqrt{6}}{3-2\sqrt{2}} \qquad \text{(lxvi. 3.)}$$

$$\therefore\ \tan 37^\circ\,30'=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$$

$$=(\sqrt{3}-\sqrt{2})(\sqrt{2}+1)$$

$$=\sqrt{6}+\sqrt{3}-\sqrt{2}-2.$$

Ex. 4. Show from the value of $\cos 22\tfrac{1}{2}^\circ$ that—

$$2\cos 5^\circ\,37'\,30''=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} \qquad \text{(liii. 4.)}$$

$2\cos 22\tfrac{1}{2}^\circ=\sqrt{2+\sqrt{2}}$. (*See* Ex. 1.)

$$2\cos^2 11^\circ\,15'=1+\cos 22\tfrac{1}{2}^\circ$$

$$=1+\frac{1}{2}\sqrt{2+\sqrt{2}}$$

$$4\cos^2 11^\circ\,15'=2+\sqrt{2+\sqrt{2}}$$

$$\therefore\ 2\cos 11^\circ\ 15' = \sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$2\cos^2 5^\circ\ 37'\ 30'' = 1 + \cos 11^\circ\ 15'$$

$$= 1 + \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$4\cos^2 5^\circ\ 37'\ 30'' = 2 + \sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$\therefore\ 2\cos 5^\circ\ 37'\ 30'' = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}.$$

Ex. 5. Given $\tan\frac{A}{2} = \frac{a}{b}$, find $\sin A$, and $\sin 2A$. (xxvi. 5.)

$$\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} = \frac{\frac{2a}{b}}{1+\frac{a^2}{b^2}} = \frac{2ab}{a^2+b^2}.$$

$$\sin 2A = 2\sin A\cos A$$

$$= \frac{4ab}{a^2+b^2} \times \frac{a^2-b^2}{a^2+b^2}$$

$$= \frac{4ab\,(a^2-b^2)}{(a^2+b^2)^2}.$$

Ex. 6. Prove that $2\cos\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}$, if A is between 270° and 360°. (xxvii. 3.)

A lies between 270° and 360°

$\frac{A}{2}$ lies between 135° and 180°

$$\left.\begin{array}{l} \sin\frac{A}{2} \text{ is } + \\ \cos\frac{A}{2} \text{ is } - \end{array}\right\} \text{and } \cos\frac{A}{2} > \sin\frac{A}{2}$$

$$\left.\begin{array}{l} \sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A} \\ \sin\frac{A}{2} - \cos\frac{A}{2} = +\sqrt{1-\sin A} \end{array}\right\}$$

$$\therefore\ 2\cos\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}.$$

Ex. 7. Given $\sin 300° = -\frac{\sqrt{3}}{2}$, find $\sin 150°$.

$$\left.\begin{array}{l}\sin 150° \text{ is } +\\ \cos 150° \text{ is } -\end{array}\right\} \text{ and } \cos 150° > \sin 150° \qquad \text{(xix. 3.)}$$

$$\left.\begin{array}{l}\sin 150° + \cos 150° = -\sqrt{1+\sin 300°}\\ \sin 150° - \cos 150° = +\sqrt{1-\sin 300°}\end{array}\right\}$$

$$\therefore\ 2\sin 150° = -\sqrt{1-\frac{\sqrt{3}}{2}}+\sqrt{1+\frac{\sqrt{3}}{2}}$$

$$= \frac{-\sqrt{2-\sqrt{3}}+\sqrt{2+\sqrt{3}}}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left\{-\frac{\sqrt{3}-1}{\sqrt{2}}+\frac{\sqrt{3}+1}{\sqrt{2}}\right\}$$

$$= \frac{1}{\sqrt{2}} \times \sqrt{2}$$

$$= 1.$$

$$\therefore\ \sin 150° = \frac{1}{2}.$$

VII.

Trig-ratios of the Angles of a Triangle.

1. $$\sin(A+B+C) = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C \quad . \quad . \quad . \quad . \quad (1)$$

$$\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \cos B \sin A \sin C - \cos C \sin A \sin B \quad . \quad . \quad . \quad . \quad (2)$$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}. \quad (3)$$

2. If A, B, C are the angles of a triangle, $A+B+C=180°$,

$$A+B = 180° - C$$

$$\therefore\ \sin(A+B) = \sin C,\quad \cos(A+B) = -\cos C.$$

Also $\frac{A+B}{2} = 90° - \frac{C}{2}$

$$\therefore\ \sin\frac{A+B}{2} = \cos\frac{C}{2},\quad \cos\frac{A+B}{2} = \sin\frac{C}{2}.$$

3. **Prop.** If $A+B+C=180°$,

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C. \quad . \quad . \qquad (4)$$

$$A+B=180°-C$$

$$\tan(A+B)=-\tan C \qquad \text{(xlvi. 6.)}$$

$$\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$

$$\therefore\ \tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

This might be deduced at once from (3) since $\tan(A+B+C)=0$,

$\therefore$ the numerator$=0$.

4. **Prop.** If $A+B+C=180°$,

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C. \quad . \qquad (5)$$

$$\begin{aligned}\text{Left-hand member} &= 2\sin(A+B)\cos(A-B)+2\sin C\cos C\\ &= 2\sin C\cos(A-B)-2\sin C\cos(A+B)\\ \text{(xlvi. 6.)}\qquad &= 2\sin C\{\cos(A-B)-\cos(A+B)\}\\ &= 4\sin A\sin B\sin C.\end{aligned}$$

5. **Prop.** If $A+B+C=180°$,

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \quad . \qquad (6)$$

$$\begin{aligned}\text{Left-hand member} &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}\\ &= 2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\cos\frac{C}{2}\cos\frac{A+B}{2}\\ &= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right\}\\ &= 4\cos\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}.\end{aligned}$$

6. **Prop.** If $A+B+C=180°$,

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \qquad (7)$$

$$\begin{aligned}\text{Left-hand member} &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}\\ &= 1+2\sin\frac{C}{2}\cos\frac{A-B}{2}-2\sin\frac{C}{2}\cos\frac{A+B}{2}\\ &= 1+2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right\}\\ &= 1+4\sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2}.\end{aligned}$$

7. PROP. If $A+B+C=180°$,

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \quad . \quad . \quad (8)$$

$$\frac{A+B}{2}=90°-\frac{C}{2}$$

$$\cot\frac{A+B}{2}=\tan\frac{C}{2} \qquad \text{(xxxii. 5.)}$$

$$\frac{\cot\frac{A}{2}\cot\frac{B}{2}-1}{\cot\frac{A}{2}+\cot\frac{B}{2}}=\frac{1}{\cot\frac{C}{2}}$$

$$\therefore \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}.$$

EXAMPLES.

Ex. 1. If the tangents of the semi-angles of a triangle are in *A.P*, show that the cosines of the whole angles will also be in *A.P*.

By question, $\tan\frac{A}{2}+\tan\frac{C}{2}=2\tan\frac{B}{2}$

$$\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}=\frac{2\sin\frac{B}{2}}{\cos\frac{B}{2}}$$

$$\frac{\sin\frac{A+C}{2}}{\cos\frac{A}{2}\cos\frac{C}{2}}=\frac{2\cos\frac{A+C}{2}}{\sin\frac{A+C}{2}}$$

$$\sin^2\frac{A+C}{2}=2\cos\frac{A}{2}\cos\frac{C}{2}\cos\frac{A+C}{2}$$

$$=\left(\cos\frac{A+C}{2}+\cos\frac{A-C}{2}\right)\cos\frac{A+C}{2}$$

$$\sin^2\frac{A+C}{2}-\cos^2\frac{A+C}{2}=\cos\frac{A+C}{2}\cos\frac{A-C}{2}$$

$$-\cos(A+C)=\frac{1}{2}(\cos A+\cos C)$$

$$2\cos B=\cos A+\cos C$$

$\therefore$ $\cos A$, $\cos B$, $\cos C$ are in *A.P*.

Ex. 2. If $A+B+C=\pi$, prove that—

$$\sin A \sin(A+2C)+\sin B\sin(B+2A)+\sin C\sin(C+2B)=0.$$

$$\text{Left-hand member}=\sin A\sin(\pi-B+C)+\sin B\sin(\pi-C+A)$$
$$+\sin C\sin(\pi-A+B)$$

$$\sin A=\sin(B+C),\qquad \sin(\pi-B+C)=\sin(\pi-(B-C))=\sin(B-C)$$

$$\text{Left-hand member}=\sin(B+C)\sin(B-C)+\sin(C+A)\sin(C-A)$$
$$+\sin(A+B)\sin(A-B)$$

(xlix. 4.)
$$=\sin^2B-\sin^2C+\sin^2C-\sin^2A+\sin^2A-\sin^2B$$
$$=0.$$

Ex. 3. If $A+B+C=\pi$, prove that—

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{A+B}{4}\sin\frac{A+C}{4}\sin\frac{B+C}{4}.$$

$$\text{Left-hand member}=2\sin\frac{A+B}{4}\cos\frac{A-B}{4}+\cos\frac{A+B}{2}\qquad\text{(lxiii. 4.)}$$

$$=2\sin\frac{A+B}{4}\cos\frac{A-B}{4}+1-2\sin^2\frac{A+B}{4}$$

$$=1+2\sin\frac{A+B}{4}\left\{\cos\frac{A-B}{4}-\sin\frac{A+B}{4}\right\}$$

$$=1+2\sin\frac{A+B}{4}\left\{\cos\frac{A-B}{4}-\cos\frac{\pi+C}{4}\right\}$$

$$\because\ \sin\frac{A+B}{4}=\sin\frac{\pi-C}{4}=\sin\left(\frac{\pi}{2}-\frac{\pi+C}{4}\right)$$

$$=\cos\frac{\pi+C}{4}$$

$$=1+2\sin\frac{A+B}{4}\,2\sin\frac{\pi+A+C-B}{8}\sin\frac{\pi+C+B-A}{8}$$

$$=1+4\sin\frac{A+B}{4}\sin\frac{2(A+C)}{8}\sin\frac{2(B+C)}{8}$$

$$=1+4\sin\frac{A+B}{4}\sin\frac{A+C}{4}\sin\frac{B+C}{4}.$$

Ex. 4. If $A+B+C=\pi$, prove that—

$$\sin\left(A+\frac{B}{2}\right)+\sin\left(B+\frac{C}{2}\right)+\sin\left(C+\frac{A}{2}\right)+1$$
$$=4\cos\frac{B-C}{4}\cos\frac{C-A}{4}\cos\frac{A-B}{4}.$$

$$\frac{B}{2}=\frac{\pi}{2}-\frac{A}{2}-\frac{C}{2},\ \therefore A+\frac{B}{2}=\frac{\pi}{2}-\frac{C-A}{2},\ \therefore \sin\left(A+\frac{B}{2}\right)=\cos\frac{C-A}{2}$$

$$\text{Left-hand member}=\cos\frac{C-A}{2}+\cos\frac{A-B}{2}+\cos\frac{B-C}{2}+1$$
$$=2\cos\frac{C-B}{4}\cos\frac{C-2A+B}{4}+2\cos^2\frac{B-C}{4}$$
$$\text{(ci. 2.)}\qquad =2\cos\frac{B-C}{4}\left\{\cos\frac{C-2A+B}{4}+\cos\frac{B-C}{4}\right\}$$
$$=2\cos\frac{B-C}{4}\,2\cos\frac{B-A}{4}\cos\frac{C-A}{4}$$
$$=4\cos\frac{B-C}{4}\cos\frac{A-B}{4}\cos\frac{C-A}{4}.$$

Ex. 5. If $A+B+C=\pi$, prove that—

$\sin 3A\sin(B-C)+\sin 3B\sin(C-A)+\sin 3C\sin(A-B)=0.$

$\sin 3A\sin(B-C)=\frac{1}{2}\{\cos(3A-B+C)-\cos(3A+B-C)\}$ (cxxiv. 2.)

and $3A-B+C=A+B+C+2(A-B)=\pi+2(A-B)$,

$$\therefore \cos(3A-B+C)=-\cos 2(A-B)$$

$$\therefore \text{Left-hand member}=\tfrac{1}{2}\{-\cos 2(A-B)+\cos 2(A-C)-\cos 2(B-C)$$
$$+\cos 2(B-A)-\cos 2(C-A)+\cos 2(C-B)\}$$
$$=0.$$

VIII.

General Values of Angles corresponding to a given Value of any Trig-ratio.

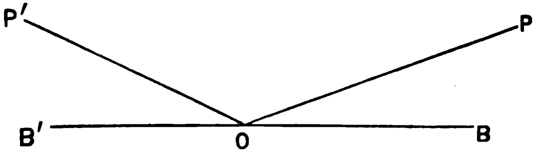
In the following important propositions, n is used to denote any integer + or − ; $2n$ represents any *even* number ; $2n+1$ or $2n-1$ represents any odd number.

$$(-1)^n=1 \text{ when } n=0,\ +1 \text{ when } n \text{ is even},\ -1 \text{ when } n \text{ is odd};$$
$$\therefore\ (-1)^n=\cos n\pi.$$

Corresponding to a given angle, there is only one value for each trig-ratio, but corresponding to a given value of any trig-ratio, there is an indefinite number of angles; and the following propositions prove that, if a is the least + angle which has a given sine or cosecant, all the angles which have the same sine or cosecant are included in the formula $n\pi + (-1)^n a$, where n is zero, or any integer + or −. For the cosine or secant the general formula is $2n\pi \pm a$, and for the tangent or cotangent $n\pi + a$.

These propositions should be carefully written out several times.

1. PROP. Find a general expression for all the angles which have a given sine or cosecant.



Let BOP, or a, be the least + angle which has the given sine (or cosecant), make $P'OB' = POB$,

Then any angle measured from OB in a + or − direction to the bounding lines OP or OP' will have the same sine (or cosecant) as the angle a.

BOP and BOP' in a + direction are a and $\pi - a$,

BOP' and BOP in a − direction are $-(\pi + a)$ and $-(2\pi - a)$,

and these angles may be increased by any number of complete revolutions, that is, by any multiple of four right angles, or $2n\pi$. Hence all the angles are included in the formulæ

$$\left.\begin{array}{l} 2n\pi + a \\ 2n\pi + \pi - a = (2n+1)\pi - a \end{array}\right\} \text{where } n \text{ is zero or any + integer,}$$

$$\left.\begin{array}{l} 2n\pi - (\pi + a) = (2n-1)\pi - a \\ 2n\pi - (2\pi - a) = (2n-2)\pi + a \end{array}\right\} \text{where } n \text{ is zero or any − integer,}$$

which are all included in the general formula

$$n\pi + (-1)^n a,$$

where n is zero or any + or − integer.

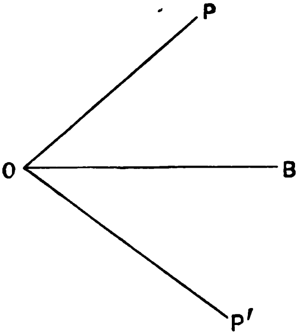
So that the solution of $\sin\theta = \sin a$, or $\operatorname{cosec}\theta = \operatorname{cosec} a$, is $\theta = n\pi + (-1)^n a$.

N.B. In choosing the general formula, it should be noticed that the coefficients of π in the four formulæ are either + or −, and either even or odd, hence we take the general coefficient n; also if the coefficient of π is even, the sign of a is $+$, and if odd, the sign is $-$, and this is expressed algebraically by $(-1)^n$.

Since $(-1)^n = \cos n\pi$, we might write the general formula—

$$n\pi + (\cos n\pi)\,a.$$

2. Prop. Find a general expression for all the angles which have a given cosine or secant.



Let BOP, or a, be the least $+$ angle which has the given cosine (or secant). Make $P'OB = POB$.

Then any angle measured from OB in a $+$ or $-$ direction to the bounding lines OP or OP' will have the same cosine (or secant) as the angle a.

BOP and BOP' in a $+$ direction are a and $2\pi - a$,

BOP' and BOP in a $-$ direction are $-a$ and $-(2\pi - a)$,

and these angles may be increased by any number of complete revolutions, that is, by any multiple of four right angles, or $2n\pi$. Hence all the angles are included in the formulæ

$$\left.\begin{array}{l}2n\pi + a\\ 2n\pi + 2\pi - a = (2n+2)\pi - a\end{array}\right\}\begin{array}{l}\text{where } n \text{ is zero or any}\\ \quad +\text{integer,}\end{array}$$

$$\left.\begin{array}{l}2n\pi - a\\ 2n\pi - (2\pi - a) = (2n-2)\pi + a\end{array}\right\}\begin{array}{l}\text{where } n \text{ is zero or any}\\ \quad -\text{integer,}\end{array}$$

which are all included in the general formula

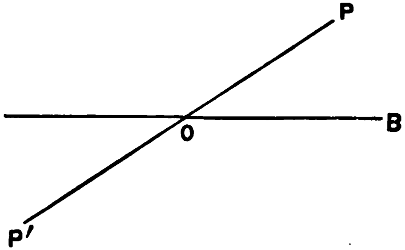
$$2n\pi \pm a$$

where n is zero, or any $+$ or $-$ integer.

So that the solution of $\cos\theta = \cos a$, or $\sec\theta = \sec a$ is $\theta = 2n\pi \pm a$

N.B.—In choosing the general formula, it should be noticed that the coefficients of π in the four formulæ are either + or −, and are all even, hence we take $2n$ as the general coefficient; and since the signs of a are + and −, we take the sign ±.

3. Prop. **Find a general expression for all the angles which have a given tangent or cotangent.**



Let BOP, or a, be the least + angle which has the given tangent (or cotangent). Produce PO to P'.

Then any angle measured from OB in a + or − direction to the bounding lines OP or OP' will have the same tangent (or cotangent) as the angle a.

BOP and BOP' in a + direction are a and $\pi+a$,

BOP' and BOP in a − direction are $-(\pi-a)$, and $-(2\pi-a)$,

and these angles may be increased by any number of complete revolutions, that is, by any multiple of four right angles, or $2n\pi$. Hence all the angles are included in the formulæ

$$\left.\begin{array}{l}2n\pi+a\\ 2n\pi+\pi+a=(2n+1)\pi+a\end{array}\right\}\begin{array}{l}\text{where } n \text{ is zero or any}\\ \quad +\text{is integer,}\end{array}$$
$$\left.\begin{array}{l}2n\pi-(\pi-a)=(2n-1)\pi+a\\ 2n\pi-(2\pi-a)=(2n-2)\pi+a\end{array}\right\}\begin{array}{l}\text{where } n \text{ is zero or any}\\ \quad -\text{integer,}\end{array}$$

which are all included in the general formula

$$n\pi+a$$

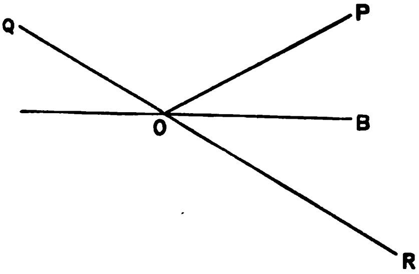
where n is zero, or any + or − integer.

So that the solution of $\tan\theta=\tan a$, or $\cot\theta=\cot a$, is $\theta=n\pi+a$.

N.B.—In choosing the general formula, it should be noticed that the coefficients of π in the four formulæ are either + or −, and either odd or even, hence we take the general coefficient n; and since all the signs of a are +, we take the sign +.

These proofs are also true, if a be any angle which has the given trig-ratio, and not necessarily the least + angle.

4. PROP. Find a general expression for all the angles which have a given sine and cosine.



Let BOP, or a, be the least + angle which has the given sine and cosine.

The bounding lines for all angles which have the given sine are OP, OQ; the bounding lines for all angles which have the given cosine are OP, OR. Hence the only bounding line for angles which have the same sine and cosine is OP, and this angle a may be increased by any multiple of four right angles.

Hence all the angles are included in the formula

$$2n\pi + a,$$

where n is zero, or any + or − integer.

IX.

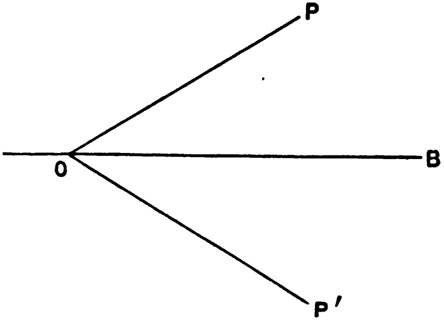
Explanation of the Double Sign in Trigonometrical Formulæ.

1. PROP. Given cos A, show *à priori* that there are two values for sin A; that is, account for the ambiguous sign in the formula

$$\sin A = \pm\sqrt{1-\cos^2 A}.$$

(i.) Geometrically.

Let BOP, or a, be the least + angle which has the given cosine.



Make $BOP' = BOP$. Then all the angles made by revolution from OB to the bounding lines OP, OP' have the given cosine,

i.e. all the angles included in $2n\pi \pm a$.

And these angles have different sines,

Hence there are *two values* for sin A.

(ii.) Analytically.

Let a be the least + angle which has the given cosine.

Then $A = 2n\pi \pm a$

$$\begin{aligned}\sin A &= \sin(2n\pi \pm a)\\ &= \sin 2n\pi \cos a \pm \cos 2n\pi \sin a\\ &= \pm \sin a, \quad \text{since } \sin 2n\pi = 0.\end{aligned}$$

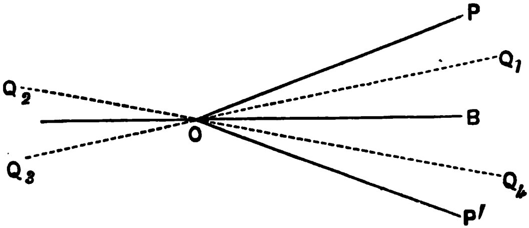
The two values of sin A are sin a and $-$ sin a.

2. Prop. Given $\cos A$ (the angle A not being known), show that there are two values each for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$; that is, account for the ambiguous signs in the formulæ

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1-\cos A}{2}}, \quad \cos \frac{A}{2} = \pm \sqrt{\frac{1+\cos A}{2}}.$$

(i.) Geometrically.

Let BOP, or a, be the least + angle which has the given cosine.



Make $BOP' = BOP$. Then all the angles made by revolution from OB to the bounding lines OP, OP' have the given cosine,

i.e. all the angles included in $2n\pi \pm a$.

Bisect these angles as in the figure, and the different values of $\frac{A}{2}$ will be bounded by the lines OQ_1, OQ_2, OQ_3, OQ_4,

i.e. all the angles included in $n\pi \pm \frac{a}{2}$.

But $\sin BOQ_2 = \sin BOQ_1$, and $\sin BOQ_3 = \sin BOQ_4$,

Hence there are only *two* values of $\sin \frac{A}{2}$.

Also $\cos BOQ_4 = \cos BOQ_1$, and $\cos BOQ_2 = \cos BOQ_3$,

Hence there are only *two* values of $\cos \frac{A}{2}$.

(ii.) Analytically.

Let a be the least + angle which has the given cosine,

Then $A=2n\pi\pm a$.

$$\therefore \sin\frac{A}{2}=\sin\left(n\pi\pm\frac{a}{2}\right)=\sin n\pi\cos\frac{a}{2}\pm\cos n\pi\sin\frac{a}{2}$$

$$=\pm\sin\frac{a}{2},\ \text{since } \sin n\pi=0,\ \cos n\pi=\pm 1.$$

The two values of $\sin\frac{A}{2}$ are $\pm\sin\frac{a}{2}$

$$\cos\frac{A}{2}=\cos\left(n\pi\pm\frac{a}{2}\right)=\cos n\pi\cos\frac{a}{2}\mp\sin n\pi\sin\frac{a}{2}$$

$$=\pm\cos\frac{a}{2}\cdot$$

The two values of $\cos\frac{A}{2}$ are $\pm\cos\frac{a}{2}\cdot$

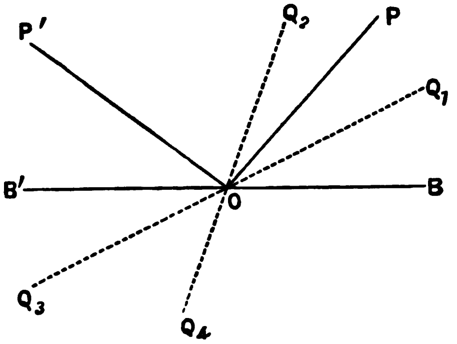
3. Prop. Given $\sin A$ (the angle A not being known), show that there are four values each for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$; that is, account for the ambiguous signs in the formulae

$$2\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A},$$

$$2\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}.$$

(i.) Geometrically.

Let BOP, or a, be the least + angle which has the given sine.



Make $B'OP' = BOP$. Then all the angles made by revolution from OB to the bounding lines OP, OP' have the given sine,

i.e. all the angles included in $n\pi + (-1)^n a$.

Bisect these angles as in the figure, and the different values of $\frac{A}{2}$ will be bounded by the lines OQ_1, OQ_2, OQ_3, OQ_4,

i.e. all the angles included in $\frac{n\pi}{2} + (-1)^n \frac{a}{2}$,

and $\sin BOQ_1$, $\sin BOQ_2$, $\sin BOQ_3$, $\sin BOQ_4$ are all different,

Hence there are *four* values for $\sin \frac{A}{2}$.

Also $\cos BOQ_1$, $\cos BOQ_2$, $\cos BOQ_3$, $\cos BOQ_4$ are all different,

Hence there are *four* values for $\cos \frac{A}{2}$.

(ii.) Analytically.

Let a be the least + angle which has the given sine,

Then $A = n\pi + (-1)^n a$.

$$\therefore \sin\frac{A}{2} = \sin\frac{1}{2}\left(n\pi + (-1)^n \frac{a}{2}\right)$$

$$= \sin\left(m\pi + \frac{a}{2}\right), \text{ if } n = 2m \text{ (even)}$$

$$= \sin m\pi \cos\frac{a}{2} + \cos m\pi \sin\frac{a}{2} \qquad \text{(vii. 3.)}$$

$$= \pm \sin\frac{a}{2}.$$

$$\sin\frac{A}{2} = \sin\frac{1}{2}\left((2m+1)\pi - \frac{a}{2}\right), \text{ if } n = 2m+1 \text{ (odd)}$$

$$= \sin\left(m\pi + \frac{\pi - a}{2}\right) = \pm \sin\frac{\pi - a}{2} = \pm \cos\frac{a}{2}.$$

The four values of $\sin\frac{A}{2}$ are $\pm\sin\frac{a}{2}$ and $\pm\cos\frac{a}{2}$.

$$\cos\frac{A}{2} = \cos\frac{1}{2}(n\pi + (-1)^n a).$$

$$= \cos\left(m\pi + \frac{a}{2}\right), \text{ if } n = 2m \text{ (even)}$$

$$= \cos m\pi \cos\frac{a}{2} - \sin m\pi \sin\frac{a}{2}.$$

$$= \pm \cos\frac{a}{2}.$$

$$\cos\frac{A}{2} = \cos\frac{1}{2}\left((2m+1)\pi - \frac{a}{2}\right), \text{ if } n = 2m+1 \text{ (odd)}$$

$$= \cos\left(m\pi + \frac{\pi - a}{2}\right)$$

$$= \pm \cos\frac{\pi - a}{2}$$

$$= \pm \sin\frac{a}{2}.$$

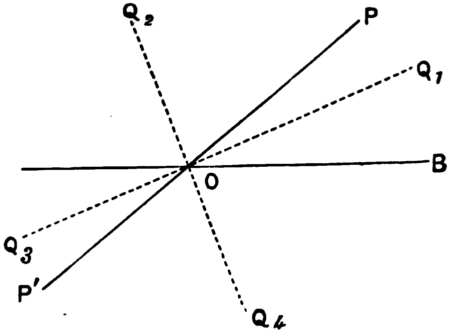
The four values of $\cos\frac{A}{2}$ are $\pm\cos\frac{a}{2}$ and $\pm\sin\frac{a}{2}$.

4. PROP. Given tan A (the angle A not being known), show that there are two values for tan $\frac{A}{2}$; that is, account for the ambiguous sign in the formula

$$\tan\frac{A}{2}=\frac{-1\pm\sqrt{1+\tan^2 A}}{\tan A}.$$

(i.) Geometrically.

Let BOP, or a, be the least + angle which has the given tangent.



Produce PO to P'. Then all the angles made by revolution from OB to the bounding lines OP, OP' have the given tangent,

***i.e.* all the angles included in $n\pi+a$.**

Bisect these angles as in the figure, and the different values of $\frac{A}{2}$ will be bounded by the lines OQ_1, OQ_2, OQ_3, OQ_4,

_i.e._ all the angles included in $\frac{n\pi}{2}+\frac{a}{2}$.

But $\tan BOQ_3=\tan BOQ_1$, and $\tan BOQ_4=\tan BOQ_2$,

Hence there are only *two* values of tan $\frac{A}{2}$.

(ii.) Analytically.

Let a be the least + angle which has the given tangent,
Then $A = n\pi + a$.

$$\therefore \tan\frac{A}{2} = \tan\frac{1}{2}(n\pi + a) = \tan\left(m\pi + \frac{a}{2}\right), \quad \text{if } n = 2m \text{ (even)}$$

$$= \frac{\tan m\pi + \tan\frac{a}{2}}{1 - \tan m\pi \tan\frac{a}{2}} = \tan\frac{a}{2}.$$

$$\tan\frac{A}{2} = \tan\frac{1}{2}\left\{(2m+1)\pi + a\right\}, \quad \text{if } n = 2m+1 \text{ (odd)}$$

$$= \tan\left(m\pi + \frac{\pi + a}{2}\right) = \tan\frac{\pi + a}{2} = -\cot\frac{a}{2}.$$

The two values of $\tan\frac{A}{2}$ are $\tan\frac{a}{2}$ and $-\cot\frac{a}{2}$.

5. **Prop.** Given $\cos A$ (the angle A not being known), show *à priori* that there are three values for $\cos\frac{A}{3}$; that is, account for the cubic equation

$$4\cos^3\frac{A}{3} - 3\cos\frac{A}{3} = \cos A \text{ to find } \cos\frac{A}{3}, \text{ when } \cos A \text{ is given.}$$

Let a be the least + angle which has the given cosine,
Then $A = 2n\pi \pm a$.

$$\therefore \cos\frac{A}{3} = \cos\frac{1}{3}(2n\pi \pm a),$$

and n is of one of the forms $3m$, $3m+1$, $3m-1$.

$$\therefore \cos\frac{A}{3} = \cos\frac{1}{3}(6m\pi \pm a), \quad \text{if } n = 3m$$

$$= \cos\left(2m\pi \pm \frac{a}{3}\right)$$

$$= \cos 2m\pi \cos\frac{a}{3} \mp \sin 2m\pi \sin\frac{a}{3}$$

$$= \cos\frac{a}{3}.$$

$$\cos\frac{A}{3}=\cos\frac{1}{3}\left\{(3m+1)\,2\pi\pm a\right\},\quad \text{if } n=3m+1$$

$$=\cos\left(2m\pi+\frac{2\pi\pm a}{3}\right)$$

$$=\cos\frac{2\pi\pm a}{3}.$$

$$\cos\frac{A}{3}=\cos\frac{1}{3}\left\{(3m-1)\,2\pi\pm a\right\},\quad \text{if } n=3m-1$$

$$=\cos\left(2m\pi-\frac{2\pi\mp a}{3}\right)$$

$$=\cos\frac{2\pi\mp a}{3}.$$

The *three* values of $\cos\frac{A}{3}$ are $\cos\frac{a}{3}$, $\cos\frac{2\pi+a}{3}$, $\cos\frac{2\pi-a}{3}$.

6. PROP. Given $\sin A$ (the angle A not being known), show *à priori* that there are three values for $\sin\frac{A}{3}$; that is, account for the cubic equation

$$4\sin^3\frac{A}{3}-3\sin\frac{A}{3}=-\sin A \text{ to find } \sin\frac{A}{3}, \text{ when } \sin A \text{ is given.}$$

Let a be the least + angle which has the given sine,

$$\text{Then } A=n\pi+(-1)^n a.$$

$$\therefore\ \sin\frac{A}{3}=\sin\frac{1}{3}\left\{n\pi+(-1)^n a\right\},$$

and n is of one of the forms $3m$, $3m+1$, $3m-1$.

$$\sin\frac{A}{3}=\sin\frac{1}{3}\left\{3m\pi+(-1)^m a\right\}, \text{ if } n=3m,$$

$$\because\ (-1)^{3m}=((-1)^3)^m=(-1)^m$$

$$=\sin\left\{(m\pi+(-1)^m\frac{a}{3}\right\}$$

$$=\cos m\pi\sin(-1)^m\frac{a}{3},\qquad \cos m\pi=(-1)^m$$

$$=(-1)^m\sin(-1)^m\frac{a}{3}\quad\begin{cases}\sin\frac{a}{3}, \text{ if } m \text{ is even},\\ -\sin\left(-\frac{a}{3}\right)=\sin\frac{a}{3}, \text{ if } m \text{ is odd}.\end{cases}$$

$$=\sin\frac{a}{3}.$$

$$\sin\frac{A}{3}=\sin\frac{1}{3}\left\{(3m+1)\pi-(-1)^m a\right\},\quad \text{if } n=3m+1$$

$$=\sin\left\{m\pi+\frac{\pi-(-1)^m a}{3}\right\}$$

$$=\cos m\pi\,\sin\frac{\pi-(-1)^m a}{3}$$

$$=(-1)^m\sin\frac{\pi-(-1)^m a}{3}$$

$$=\sin\frac{\pi-a}{3},\ \text{or}\ -\sin\frac{\pi+a}{3},\ \text{as } m \text{ is even or odd.}$$

$$\sin\frac{A}{3}=\sin\frac{1}{3}\left\{(3m-1)\pi-(-1)^m a\right\},\quad \text{if } n=3m-1$$

$$=\sin\left\{m\pi-\frac{\pi+(-1)^m a}{3}\right\}$$

$$=-\cos m\pi\,\sin\frac{\pi+(-1)^m a}{3}$$

$$=-(-1)^m\sin\frac{\pi+(-1)^m a}{3}$$

$$=-\sin\frac{\pi+a}{3},\ \text{or}\ \sin\frac{\pi-a}{3},\ \text{as } m \text{ is even or odd.}$$

the *three* values of $\sin\frac{A}{3}$ are $\sin\frac{a}{3},\quad \sin\frac{\pi-a}{3},\quad -\sin\frac{\pi+a}{3}$.

X.

Trig-Ratios of 18°, 36°, 9°, etc.

1. **Prop.** Find sin 18°.

(i.) Analytically.

$$\text{Since } 54^\circ=90^\circ-36^\circ$$

$$\cos 54^\circ=\sin 36^\circ$$

Let $A=18^\circ$, $\therefore\ \cos 3A=\sin 2A$,

$$4\cos^3 A-3\cos A=2\sin A\cos A$$

$$4\cos^2 A-3=2\sin A,\quad \text{since } \cos A \text{ is not zero.}$$

$$4(1-\sin^2 A)-3-2\sin A=0$$

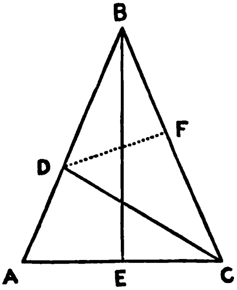
$$\sin^2 A+\frac{1}{2}\sin A+\left(\frac{1}{4}\right)^2=\frac{1}{4}+\frac{1}{16}=\frac{5}{16}$$

$$\sin A=\frac{-1\pm\sqrt{5}}{4}.$$

Since sin 18° is +, we take the upper sign,

$$\therefore\ \sin 18^\circ=\frac{\sqrt{5}-1}{4}=\cos 72^\circ.$$

(ii.) Geometrically.



Let ABC be a triangle having each of the angles at the base double of the angle at the vertex. Make the construction as in Euc. iv. 10.

If the vertical angle $=A$, then

$A+2A+2A=180^\circ$, $\therefore A=36^\circ$, and $ABE=18^\circ$

Denote AE by m linear units, and AB by n linear units,

Then $BD=AC=2m$, and $BA.AD=BD^2$

$$\therefore\ n(n-2m)=(2m)^2$$

$$n^2-2mn-4m^2=0$$

$$\left(\frac{m}{n}\right)+\frac{1}{2}\left(\frac{m}{n}\right)=\frac{1}{4}$$

from which $\frac{m}{n}=\frac{\sqrt{5}-1}{4}=\sin 18^\circ.$

Draw DF perpendicular to BC, then since $DB=DC$, DF bisects BC;

$$\therefore\ \cos 36^\circ=\cos DBF=\frac{BF}{BD}=\frac{\frac{n}{2}}{2m}=\frac{n}{4m}=\frac{1}{\sqrt{5}-1}=\frac{\sqrt{5}+1}{4}=\sin 54^\circ.$$

2. Since $\cos^2 18° = 1 - \sin^2 18° = 1 - \left\{\frac{6-2\sqrt{5}}{16}\right\} = \frac{10+2\sqrt{5}}{16}$

$$\therefore \cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°.$$

$$2\cos^2 36° = 1 + \cos 72° = 1 + \sin 18° = 1 + \frac{\sqrt{5}-1}{4}$$

$$= \frac{3+\sqrt{5}}{4}$$

$$\cos^2 36° = \frac{3+\sqrt{5}}{8} = \frac{6+2\sqrt{5}}{16}$$

$$\therefore \cos 36° = \frac{\sqrt{5}+1}{4} = \sin 54°.$$

$$\sin^2 36° = 1 - \cos^2 36° = 1 - \frac{6+2\sqrt{5}}{16} = \frac{10-2\sqrt{5}}{16}$$

$$\therefore \sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°.$$

3. Show *à priori* why two values are found for sin 18°.

We had $\sin 2A = \cos 3A$

$$\cos\left(\frac{\pi}{2} - 2A\right) = \cos 3A$$

$$\frac{\pi}{2} - 2A = 2n\pi \pm 3A$$

$$(2 \pm 3)A = \frac{\pi}{2} - 2n\pi = \frac{(1-4n)\pi}{2}$$

$$\therefore A = \frac{(1-4n)\pi}{2(2\pm 3)},$$

$$\text{if } n = 0,\ A = \frac{\pi}{10} \quad \text{or} -\frac{\pi}{2},$$

$$\text{if } n = 1,\ A = -\frac{3\pi}{10} \quad \text{or } \frac{3\pi}{2},$$

$$\text{and } \cos\frac{3\pi}{2} = \cos\left(-\frac{\pi}{2}\right) = 0,$$

which accounts for the factor $\cos A$ in the equation ;

$\sin\left(-\frac{3\pi}{10}\right) = -\cos\frac{\pi}{5} = \frac{-1-\sqrt{5}}{4}$, the other root of the equation.

4. **Prop.** Having found the sines and cosines of 18°, 36°, 54°, 72°, we can now find sin 9° and cos 9°.

$$\sin 9^\circ + \cos 9^\circ = +\sqrt{1+\sin 18^\circ}$$

$$\sin 9^\circ - \cos 9^\circ = -\sqrt{1-\sin 18^\circ}$$

since sin 9° and cos 9° are positive, and cos 9° > sin 9°,

$$\sin 9^\circ = \frac{1}{2}\left\{\sqrt{1+\sin 18^\circ} - \sqrt{1-\sin 18^\circ}\right\}$$

$$\cos 9^\circ = \frac{1}{2}\left\{\sqrt{1+\sin 18^\circ} + \sqrt{1-\sin 18^\circ}\right\} \qquad \text{(xcvi. 2.)}$$

$$\therefore\ \sin 9^\circ = \frac{1}{4}\left\{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\right\} = \cos 81^\circ.$$

$$\cos 9^\circ = \frac{1}{4}\left\{\sqrt{3+\sqrt{5}} + \sqrt{5+\sqrt{5}}\right\} = \sin 81^\circ.$$

We can then find $\sin 3^\circ = \sin(18^\circ - 15^\circ) = \cos 87^\circ$, and hence we know the sines and cosines of 6°, 12°, etc.

5. $$\text{Tan } 36^\circ = \sqrt{\sec^2 36^\circ - 1} = \sqrt{(\sqrt{5}-1)^2 - 1} = \sqrt{5-2\sqrt{5}}$$

$$= \sqrt{5-\sqrt{20}} = \sqrt{5 - 4\cdot47213595} = \cdot52786405 = \cdot7265\ \ldots$$

correct to four places of decimals.

6. $$\text{Tan } 27^\circ = \frac{1-\cos 54^\circ}{\sin 54^\circ} = \frac{1-\sin 36^\circ}{\cos 36^\circ} = \frac{1-\frac{1}{4}\sqrt{10-2\sqrt{5}}}{\frac{\sqrt{5}+1}{4}}$$

$$= \sqrt{5} - 1 - \frac{1}{4}\sqrt{(10-2\sqrt{5})\,(\sqrt{5}-1)^2}$$

$$= \sqrt{5} - 1 - \sqrt{5-2\sqrt{5}}.$$

7. Show that sin 9° lies between ·156 and ·157.

$$\sin 9^\circ = \frac{1}{4}\left\{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\right\} \qquad \text{(lxxxi. 2.)}$$

$$= \frac{1}{4}\left\{\sqrt{5\cdot2360680} - \sqrt{2\cdot7639320}\right\}$$

$$= \frac{1}{4}\left\{2\cdot288\ \ldots - 1\cdot662\ \ldots\right\}$$

$$= \frac{1}{4}\left\{\cdot626\ \ldots\right\}$$

$$= \cdot156\ \ldots$$

XI.

Identities, Equations, Elimination.

1. A Trigonometrical Identity is the equality of two expressions involving Trig-ratios, which is true for all values of the angles involved: thus $\sec^2 A = 1 + \tan^2 A$, $\sin^2 A - \cos^2 B = \sin^2 B - \cos^2 A$ are identities.

In proving Identities, it is advisable to begin with the most complicated side, and work towards the other. In some few cases it may be necessary to reduce both sides to the same expression.

The original Identity may often be simplified by transposition.

EXAMPLES.

Ex. 1. Prove that—

$$\sin(B-C) + \sin(C-A) + \sin(A-B) + 4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2} = 0.$$

Left-hand member—

$$= 2\sin\frac{B-A}{2}\cos\frac{B-2C+A}{2} + 2\sin\frac{A-B}{2}\cos\frac{A-B}{2}$$

$$+ 4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$$

$$= 2\sin\frac{B-A}{2}\left\{\cos\frac{B-2C+A}{2} - \cos\frac{A-B}{2} - 2\sin\frac{B-C}{2}\sin\frac{C-A}{2}\right\}$$

$$= 2\sin\frac{B-A}{2}\left\{2\sin\frac{A-C}{2}\sin\frac{C-B}{2} - 2\sin\frac{B-C}{2}\sin\frac{C-A}{2}\right\}$$

$$= 2\sin\frac{B-A}{2}\left\{2\sin\frac{C-A}{2}\sin\frac{B-C}{2} - 2\sin\frac{B-C}{2}\sin\frac{C-A}{2}\right\}$$

$$= 0.$$

Ex. 2. Prove $2 \operatorname{cosec} 4A + 2 \cot 4A = \cot A - \tan A$.

$$\text{Left-hand member} = 2\,\frac{1 + \cos 4A}{\sin 4A} \qquad (\text{v. } 2.)$$

$$= \frac{4 \cos^2 2A}{2 \sin 2A \cos 2A}$$

$$= \frac{2 \cos 2A}{\sin 2A}$$

$$= \frac{2 (\cos^2 A - \sin^2 A)}{2 \sin A \cos A}$$

$$= \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A}$$

$$= \cot A - \tan A.$$

Ex. 3. Prove that $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \tan 2A + \sec 2A$.

$$\text{Right-hand member} = \frac{\sin 2A + 1}{\cos 2A}$$

$$= \frac{2 \sin A \cos A + \sin^2 A + \cos^2 A}{\cos^2 A - \sin^2 A} \qquad (\text{cxxviii. } 2.)$$

$$= \frac{(\cos A + \sin A)^2}{(\cos A + \sin A)(\cos A - \sin A)}$$

$$= \frac{\cos A + \sin A}{\cos A - \sin A}.$$

Ex. 4. Prove that $\dfrac{\sin 3A}{\sin A} - \dfrac{\cos 3A}{\cos A} = 2$.

$$\text{Left-hand member} = \frac{\sin 3A \cos A - \cos 3A \sin A}{\sin A \cos A} \qquad (\text{xxiv. } 4.)$$

$$= \frac{\sin (3A - A)}{\sin A \cos A} = \frac{\sin 2A}{\sin A \cos A} = 2.$$

Or, left-hand member $= \dfrac{3 \sin A - 4 \sin^3 A}{\sin A} - \dfrac{4 \cos^3 A - 3 \cos A}{\cos A}$

$$= 3 - 4 \sin^2 A - 4 \cos^2 A + 3$$

$$= 6 - 4 (\sin^2 A + \cos^2 A) = 6 - 4 = 2.$$

Ex. 5. Prove that $\dfrac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} - \dfrac{\cos A - \cos 3A}{\sin 3A - \sin A} = \dfrac{\sin A}{\cos 2A \cos 3A}$.

$$\text{Left-hand member} = \frac{2 \sin 3A \sin A}{2 \cos 3A \sin A} - \frac{2 \sin 2A \sin A}{2 \cos 2A \sin A}$$

$$= \frac{\sin 3A}{\cos 3A} - \frac{\sin 2A}{\cos 2A} \qquad \text{(viii. 2.)}$$

$$= \frac{\sin 3A \cos 2A - \cos 3A \sin 2A}{\cos 2A \cos 3A}$$

$$= \frac{\sin (3A - 2A)}{\cos 2A \cos 3A}$$

$$= \frac{\sin A}{\cos 2A \cos 3A}.$$

Ex. 6. Prove that $2 \sec A = \tan\left(\dfrac{\pi}{4} + \dfrac{A}{2}\right) + \cot\left(\dfrac{\pi}{4} + \dfrac{A}{2}\right)$

$$\text{since} \quad \tan A + \cot A = \frac{2}{\sin 2A},$$

$$\therefore \text{right-hand member} = \frac{2}{\sin\left(\dfrac{\pi}{2} + A\right)} = \frac{2}{\cos A} = 2 \sec A.$$

$$\textit{Or}, \text{ right-hand member} = \frac{1 + \tan\dfrac{A}{2}}{1 - \tan\dfrac{A}{2}} + \frac{1 - \tan\dfrac{A}{2}}{1 + \tan\dfrac{A}{2}} \qquad \text{(cxxviii. 2.)}$$

$$= \frac{1 + 2 \tan\dfrac{A}{2} + \tan^2\dfrac{A}{2} + 1 - 2 \tan\dfrac{A}{2} + \tan^2\dfrac{A}{2}}{1 - \tan^2\dfrac{A}{2}}$$

$$= \frac{2\left(1 + \tan^2\dfrac{A}{2}\right)}{1 - \tan^2\dfrac{A}{2}} = \frac{2\left(\cos^2\dfrac{A}{2} + \sin^2\dfrac{A}{2}\right)}{\cos^2\dfrac{A}{2} - \sin^2\dfrac{A}{2}} = \frac{2}{\cos A} = 2 \sec A.$$

Ex. 7. Prove that—

$$(2\cos A+1)(2\cos A-1)(2\cos 2A-1)=2\cos 4A+1.$$

$$\begin{aligned}\text{Left-hand member} &= (4\cos^2 A-1)(2\cos 2A-1)\\ &=\{2(1+\cos 2A)-1\}\{2\cos 2A-1\}\\ &=(2\cos 2A+1)(2\cos 2A-1) && \text{(xviii. 6.}\\ &=4\cos^2 2A-1\\ &=2(\cos 4A+1)-1\\ &=2\cos 4A+1.\end{aligned}$$

Ex. 8. Prove that $\tan 3A\tan 2A\tan A=\tan 3A-\tan 2A-\tan A$.

$$\begin{aligned}\text{Tan } 3A &= \tan(2A+A)\\ &=\frac{\tan 2A+\tan A}{1-\tan 2A\tan A} && \text{(xxxiv. 4.)}\end{aligned}$$

$$\tan 3A-\tan 3A\tan 2A\tan A=\tan 2A+\tan A$$

$$\therefore \tan 3A-\tan 2A-\tan A=\tan 3A\tan 2A\tan A.$$

Similarly (xcvi. 1) $\tan 5A-\tan 3A-\tan 2A=\tan 5A\tan 3A\tan 2A$.

2. **A Trigonometrical Equation is the equality of two expressions involving Trig-ratios, which equality is true only for certain values of the unknown angle involved.**

To solve the equation, we have to find the values of the unknown angle which satisfy the equation: **and in all cases the solution should be given in the general form which includes all the angles which satisfy the equation.**

Several different solutions may sometimes be found which differ in *form* only, and are in all cases equivalent. [*See* Ex. 3 and 5.]

EXAMPLES.

Ex. 1. Write down formulæ including all angles which satisfy—

(i.) $2\sin A=1$, (ii.) $2\sin^2 A=1$.

$$\text{(i.)}\ \sin A=\frac{1}{2}$$

$$A=\frac{\pi}{6}=n\pi+(-1)^n\frac{\pi}{6}.$$

$$\text{(ii.)}\ \sin^2 A=\frac{1}{2}\qquad\text{(xxiii. 2.)}$$

$$\sin A=\pm\frac{1}{\sqrt{2}}$$

$$A=\frac{\pi}{4},\quad\text{or}-\frac{\pi}{4}$$

$$=n\pi+(-1)^n\frac{\pi}{4},\quad\text{or } n\pi-(-1)^n\frac{\pi}{4}$$

$$=n\pi\pm\frac{\pi}{4}.$$

Ex. 2. Write down the general value of A if $\sec^2 A=\frac{4}{3}$.

$$\sec A=\pm\frac{2}{\sqrt{3}}$$

$$A=\frac{\pi}{6},\quad\text{or } \pi-\frac{\pi}{6}\qquad\text{(xxxii. 2.)}$$

$$=2n\pi\pm\frac{\pi}{6},\quad\text{or } 2n\pi\pm\left(\pi-\frac{\pi}{6}\right)=(2n\pm1)\pi\mp\frac{\pi}{6}$$

$$A=n\pi\pm\frac{\pi}{6}.$$

Ex. 3. Find the general values of A, when $\cos 2A=\sin A$.

$$\text{(i.)}\ \cos 2A=\cos\left(\frac{\pi}{2}-A\right)$$

$$\therefore\ \frac{\pi}{2}-A=2n\pi\pm2A.\qquad\text{(xvi. 5.)}$$

$$\textit{Or}\quad \text{(ii.)}\ \sin\left(\frac{\pi}{2}-2A\right)=\sin A$$

$$\therefore\ \frac{\pi}{2}-2A=n\pi+(-1)^n A.$$

Ex. 4. Find all the positive values of A less than two right angles which satisfy the equation $\sin 5A=\sin 11A$.

$$\sin 11A=\sin 5A \qquad \text{(lxviii. 2.)}$$

$$\therefore\ 11A=n\pi+(-1)^n 5A$$

$$n=0,\ \text{gives } 11A=5A, \qquad \text{or } A=0.$$

$$n=1,\ \text{gives } 11A=\pi-5A, \qquad \therefore\ A=\frac{\pi}{16}.$$

$$n=2,\ \text{gives } 11A=2\pi+5A, \qquad \therefore\ A=\frac{\pi}{3}.$$

etc. etc.

Ex. 5. Solve the equation $\sin\theta+\cos\theta=\sqrt{2}$.

$$\text{(i.)}\ \sin\theta+\cos\theta=\sqrt{2} \qquad \text{(xxvii. 6.}$$

$$\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta=2$$

$$1+\sin 2\theta=2$$

$$\sin 2\theta=1$$

$$\therefore\ 2\theta=\frac{\pi}{2}=n\pi+(-1)^n\frac{\pi}{2}.$$

$$\textit{Or}\quad \text{(ii.)}\ \sin\theta+\cos\theta=\sqrt{2}$$

$$\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta=1$$

$$\cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}=1$$

$$\cos\left(\theta-\frac{\pi}{4}\right)=1$$

$$\therefore\ \theta-\frac{\pi}{4}=0=2n\pi.$$

Or (iii.) $\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta=1$

$$\sin\frac{\pi}{4}\cos\theta+\cos\frac{\pi}{4}\sin\theta=1$$

$$\sin\left(\frac{\pi}{4}+\theta\right)=1$$

$$\therefore\ \frac{\pi}{4}+\theta=\frac{\pi}{2}$$

$$=n\pi+(-1)^n\frac{\pi}{2}.$$

Ex. 6. Solve the equation $\operatorname{cosec}\theta+\cot\theta=\sqrt{3}$.

$$\frac{1+\cos\theta}{\sin\theta}=\sqrt{3}$$ (xlii. 6.)

$$\cot\frac{\theta}{2}=\sqrt{3}$$

$$\therefore\ \frac{\theta}{2}=\frac{\pi}{6}=n\pi+\frac{\pi}{6}.$$

Ex. 7. Solve the equation $\sin\frac{\theta}{2}=\operatorname{cosec}\theta-\cot\theta$.

$$\sin\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$$ (liii. 6.)

$$=\tan\frac{\theta}{2}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$\sin\frac{\theta}{2}=0,\quad\text{or }\cos\frac{\theta}{2}=1$$

$$\therefore\ \frac{\theta}{2}=0=n\pi,\quad\text{or }\frac{\theta}{2}=0=2n\pi.$$

Ex. 8. Solve the equation $\sin 5\theta\cos 3\theta=\sin 9\theta\cos 7\theta$.

$$\frac{1}{2}\left\{\sin(5\theta+3\theta)+\sin(5\theta-3\theta)\right\}=\frac{1}{2}\left\{\sin(9\theta+7\theta)+\sin(9\theta-7\theta)\right\}$$

$$\sin 8\theta+\sin 2\theta=\sin 16\theta+\sin 2\theta$$ (lxxxi. 5.)

$$\sin 16\theta=\sin 8\theta$$

$$\therefore\ 16\theta=n\pi+(-1)^n 8\theta.$$

Ex. 9. Find all the values of θ which satisfy the equation

$$\cos\theta\cos 3\theta = \cos 2\theta\cos 6\theta.$$

$$\frac{1}{2}\left\{\cos(3\theta+\theta)+\cos(3\theta-\theta)\right\} = \frac{1}{2}\left\{\cos(6\theta+2\theta)+\cos(6\theta-2\theta)\right\}$$

$$\cos 4\theta + \cos 2\theta = \cos 8\theta + \cos 4\theta$$

$$\cos 8\theta = \cos 2\theta \qquad \text{(v. 3.)}$$

$$\therefore\ 8\theta = 2n\pi \pm 2\theta.$$

$$\text{or}\quad 4\theta = n\pi \pm \theta.$$

Ex. 10. Explain why it is that exactly the same series of angles are given by the two equations $\theta+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{6}$, and $\theta-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{3}$.

These results have been obtained from—

$$\sin\left(\theta+\frac{\pi}{4}\right)=\sin\frac{\pi}{6},\ \text{and}\ \cos\left(\theta-\frac{\pi}{4}\right)=\cos\frac{\pi}{3}$$

$$\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}=\frac{1}{2},\ \text{and}\ \cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}=\frac{1}{2}$$

$$\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)=\frac{1}{2},\ \text{and}\ \frac{1}{\sqrt{2}}(\cos\theta+\sin\theta)=\frac{1}{2} \qquad \text{(xxxvi. 6.)}$$

$$\therefore\ \sin\theta+\cos\theta=\frac{1}{\sqrt{2}}.$$

The results are different *forms* of the solution of this equation.

Ex. 11. Solve the equation $\sin^2 n\theta - \sin(n-1)\theta = \sin^2\theta$.

$$\sin^2 n\theta - \sin^2\theta = \sin(n-1)\theta$$

$$\sin(n+1)\theta\sin(n-1)\theta = \sin(n-1)\theta \qquad \text{(lxix. 1.)}$$

$$\sin(n-1)\theta = 0,\quad \text{or}\quad \sin(n+1)\theta = 1$$

$$\therefore\ (n-1)\theta = m\pi,\quad \text{or}\quad (n+1)\theta = m\pi + (-1)^m\frac{\pi}{2}.$$

Ex. 12. Solve the equation $\sin\frac{n+1}{2}\theta + \sin\frac{n-1}{2}\theta = \sin\theta$.

$$2\sin\frac{n\theta}{2}\cos\frac{\theta}{2} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \qquad \text{(cxx. 1.)}$$

$$\cos\frac{\theta}{2}=0,\quad \text{or}\quad \sin\frac{n\theta}{2}=\sin\frac{\theta}{2}$$

$$\therefore\ \frac{\theta}{2}=2m\pi\pm\frac{\pi}{2},\quad \text{or}\quad \frac{n\theta}{2}=m\pi+(-1)^m\frac{\theta}{2}.$$

Ex. 13. If $\tan(\pi\cot\theta)=\cot(\pi\tan\theta)$, find all the values of θ.

$$\tan(\pi\cot\theta)=\tan\left(\frac{\pi}{2}-\pi\tan\theta\right)$$

$$\pi\cot\theta=n\pi+\frac{\pi}{2}-\pi\tan\theta \qquad \text{(ci. 1.)}$$

$$\frac{1}{\tan\theta}=\frac{2n+1}{2}-\tan\theta$$

$$\tan^2\theta-\frac{2n+1}{2}\tan\theta+\left(\frac{2n+1}{4}\right)^2=\frac{4n^2+4n+1}{16}-1=\frac{4n^2+4n-15}{16}$$

$$\tan\theta=\frac{2n+1}{4}\pm\frac{\sqrt{4n^2+4n-15}}{4}$$

$$\therefore\ \theta=\tan^{-1}\left\{\frac{2n+1\pm\sqrt{4n^2+4n-15}}{4}\right\}.$$

3. Elimination in Trigonometrical Equations.

As in Algebra, several equations may be given, by the combination of which certain quantities may be eliminated, and a result obtained involving the remaining quantities.

The relations existing between the Trig-ratios are important reducing elements—

$$\sin^2 A+\cos^2 A=1,\quad \sec^2 A=1+\tan^2 A,\quad \tan A=\frac{1}{\cot A},\ \text{etc.}$$

e.g. If it is required to eliminate θ between the equations,

$$a\cos\theta+b\sin\theta=c,\quad p\cos\theta+q\sin\theta=r,$$

we can solve the equations to obtain the values of $\cos\theta$ and $\sin\theta$, and then substitute the results in

$$\cos^2\theta+\sin^2\theta=1.$$

Again in

$$a\sin\theta+b\cos\theta=c,\ \text{and}\ a\cos\theta-b\sin\theta=d,$$

if we square each equation and add, we obtain at once $a^2+b^2=c^2+d^2$.

EXAMPLES.

Ex. 1. Eliminate θ between the equations—

$$m = \operatorname{cosec}\theta - \sin\theta, \quad n = \sec\theta - \cos\theta.$$

$$m = \frac{1-\sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta} \qquad \text{(xiii. 8.)}$$

$$n = \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$$

$$\therefore\ m^2n = \cos^3\theta, \quad mn^2 = \sin^3\theta$$

$$m^{\frac{2}{3}}n^{\frac{1}{3}} = \cos\theta, \quad m^{\frac{1}{3}}n^{\frac{2}{3}} = \sin\theta$$

substitute in $\sin^2\theta + \cos^2\theta = 1$

$$m^{\frac{2}{3}}n^{\frac{4}{3}} + m^{\frac{4}{3}}n^{\frac{2}{3}} = 1$$

$$\therefore\ m^{\frac{2}{3}}n^{\frac{2}{3}}\left(m^{\frac{2}{3}} + n^{\frac{2}{3}}\right) = 1.$$

Ex. 2. Eliminate x and y between the equations—

$$\tan x + \tan y = a, \quad \cot x + \cot y = b, \quad x + y = c.$$

$$\frac{1}{\tan x} + \frac{1}{\tan y} = b \qquad \text{(cxviii. 2.)}$$

$$\frac{\tan x + \tan y}{\tan x \tan y} = b, \quad \frac{a}{\tan x \tan y} = b, \quad \tan x \tan y = \frac{a}{b}$$

$$\tan(x+y) = \tan c$$

$$\frac{\tan x + \tan y}{1 - \tan x \tan y} = \tan c$$

$$\frac{a}{1-\frac{a}{b}} = \tan c = \frac{ab}{b-a}$$

$$\therefore\ \cot c = \frac{b-a}{ab} = \frac{1}{a} - \frac{1}{b}.$$

Ex. 3. Eliminate θ between $\left.\begin{array}{r} x=2a\cos\theta\cos 2\theta-a\cos\theta, \\ y=2a\cos\theta\sin 2\theta-a\sin\theta \end{array}\right\}$

$$\left.\begin{array}{l} \dfrac{x}{a}=2\cos\theta\cos 2\theta-\cos\theta \\[2ex] \dfrac{y}{a}=2\cos\theta\sin 2\theta-\sin\theta \end{array}\right\} \quad . \qquad \text{(xlviii. 6.)}$$

$$\left.\begin{array}{l} \dfrac{x^2}{a^2}=4\cos^2\theta\cos^2 2\theta+\cos^2\theta-4\cos^2\theta\cos 2\theta \\[2ex] \dfrac{y^2}{a^2}=4\cos^2\theta\sin^2 2\theta+\sin^2\theta-4\cos\theta\sin\theta\sin 2\theta \end{array}\right\} \quad .$$

$$\begin{aligned} \frac{x^2}{a^2}+\frac{y^2}{a^2}&=4\cos^2\theta\,(\cos^2 2\theta+\sin^2 2\theta)+(\cos^2\theta+\sin^2\theta) \\ &\qquad -4\cos\theta\,[\cos 2\theta\cos\theta-\sin 2\theta\sin\theta] \\ &=4\cos^2\theta+1-4\cos^2\theta \\ &=1 \end{aligned}$$

$\therefore\ x^2+y^2=a^2.$

Ex. 4. Eliminate a, b, c, from the three equations,

$$a=b\cos C+c\cos B,\quad b=a\cos C+c\cos A,\quad c=a\cos B+b\cos A$$

which subsist between the sides a, b, c, and the angles A, B, C of a plane triangle.

$$b\cos C+c\cos B=a \quad . \quad . \quad . \quad . \quad . \qquad \text{(i.)}$$

$$b-c\cos A=a\cos C$$

$$\therefore\ b\cos C-c\cos A\cos C=a\cos^2 C \quad . \quad . \quad . \quad . \qquad \text{(ii.)}$$

(i.)-(ii.) . . $c\,[\cos B+\cos A\cos C]=a\,(1-\cos^2 C)=a\sin^2 C$

Similarly $\quad a\,[\cos B+\cos A\cos C]=c\sin^2 A \qquad$ (cxix. 1.)

$$\frac{\sin^2 C}{\cos B+\cos A\cos C}=\frac{\cos B+\cos A\cos C}{\sin^2 A}$$

$$\sin^2 A\sin^2 C=(\cos B+\cos A\cos C)^2$$

$$(1-\cos^2 A)\,(1-\cos^2 C)=\cos^2 B+2\cos A\cos B\cos C+\cos^2 A\cos^2 C$$

$$1-\cos^2 A-\cos^2 C+\cos^2 A\cos^2 C=\cos^2 B+2\cos A\cos B\cos C+\cos^2 A\cos^2 C$$

$$\therefore\ \cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$$

Ex. 5. Eliminate θ between the equations—

$$\left.\begin{aligned} x&=a\cos\theta+b\cos 2\theta,\\ y&=a\sin\theta+b\sin 2\theta.\end{aligned}\right\}\qquad\text{(lii. 5.)}$$

$$\begin{aligned} x\cos\theta+y\sin\theta&=a\,(\cos^2\theta+\sin^2\theta)+b\cos(2\theta-\theta)\\ &=a+b\cos\theta\end{aligned}$$

$$(x-b)\cos\theta+y\sin\theta=a\quad.\;.\;.\;.\;.\;.\;.\quad\text{(i.)}$$

$$\begin{aligned} x\sin\theta-y\cos\theta&=b\,(\cos 2\theta\sin\theta-\sin 2\theta\cos\theta)=-b\sin(2\theta-\theta)\\ &=-b\sin\theta\end{aligned}$$

$$(x+b)\sin\theta-y\cos\theta=0\quad.\;.\;.\;.\;.\;.\;.\;.\;.\quad\text{(ii.)}$$

from (i.) and (ii.) $\cos\theta=\dfrac{a\,(x+b)}{x^2+y^2-b^2}$

$$\sin\theta=\frac{ay}{x^2+y^2-b^2}$$

substitute in $\cos^2\theta+\sin^2\theta=1$

$$\therefore\ a^2\,\{(x+b)^2+y^2\}=\{x^2+y^2-b^2\}^2.$$

XII.

Inverse Trigonometrical Ratios.

1. If $\sin A=x$, then $A=\sin^{-1}x$, where $\sin^{-1}x$ is an abbreviation for "the angle whose sine is x."

Be careful to note that -1 is not an index,

$$\sin^{-1}x \text{ is not } (\sin x)^{-1}, \text{ which}=\frac{1}{\sin x}.$$

Note also that $\sin^{-1}\dfrac{\sqrt{3}}{2}=\cos^{-1}\dfrac{1}{2}$ is not an identity, but is true only for the particular angle 60°.

$$\sin^{-1}\frac{\sqrt{3}}{2} \text{ is one value of } n\pi+(-1)^n\frac{\pi}{3},$$

$$\text{and } \cos^{-1}\frac{1}{2} \text{ is one value of } 2n\pi\pm\frac{\pi}{3}.$$

$$\begin{aligned} &\text{if } \sin^{-1}x=A,\quad x=\sin(n\pi+(-1)^nA)\\ &\text{if } \cos^{-1}x=A,\quad x=\cos(2n\pi\pm A)\\ &\text{if } \tan^{-1}x=A,\quad x=\tan(n\pi+A).\end{aligned}$$

2. PROP. Prove—

$$\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right\}$$

Let $\sin^{-1}x=A$, then $\sin A=x$, $\therefore \cos A=\sqrt{1-x^2}$

$\sin^{-1}y=B$, then $\sin B=y$, $\therefore \cos B=\sqrt{1-y^2}$

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$

$$=x\sqrt{1-y^2}\pm\sqrt{1-x^2}\,.\,y$$

$$A\pm B=\sin^{-1}\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\}$$

$$\therefore \sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\}.$$

By a similar proof, $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\{xy\mp\sqrt{(1-x^2)(1-y^2)}\}$.

3. PROP. Prove $\tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\dfrac{x\pm y}{1\mp xy}$.

Let $\tan^{-1}x=A$, $\therefore \tan A=x$

$\tan^{-1}y=B$, $\therefore \tan B=y$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

$$=\frac{x\pm y}{1\mp xy}$$

$$A\pm B=\tan^{-1}\frac{x\pm y}{1\mp xy}$$

$$\therefore \tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\frac{x\pm y}{1\mp xy}.$$

4. Any relations which have been established among the trig-ratios may be expressed by means of the inverse notation.

$$\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\operatorname{cosec}^{-1}\frac{1}{x}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}.$$

$$\tan^{-1}x=\cot^{-1}\frac{1}{x}=\sin^{-1}\frac{x}{\sqrt{1+x^2}}=\cos^{-1}\frac{1}{\sqrt{1+x^2}}.$$

$$\cos 2A = 2\cos^2 A - 1, \quad \therefore 2\cos^{-1} x = \cos^{-1}(2x^2 - 1).$$

$$\sin 2A = 2\sin A\cos A \quad \therefore 2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2}).$$

$$\cot(A \pm B) = \frac{\cot A\cot B \mp 1}{\cot A \pm \cot B} \quad \therefore \cot^{-1} x \pm \cot^{-1} y = \cot^{-1}\left\{\frac{xy \mp 1}{x \pm y}\right\}.$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} \quad \therefore 2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}.$$

$$\sin 3A = 3\sin A - 4\sin^3 A \quad \therefore 3\sin^{-1} x = \sin^{-1}(3x - 4x^3).$$

$$\cos 3A = 4\cos^3 A - 3\cos A \quad \therefore 3\cos^{-1} x = \cos^{-1}(4x^3 - 3x).$$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan B\tan C - \tan C\tan A - \tan A\tan B}.$$

$$\therefore \tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-xz}.$$

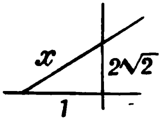
EXAMPLES.

Ex. 1. Show that $\sec^{-1} 3 = 2\cot^{-1}\sqrt{2}$. (iv. 4.)

$$2\cot^{-1}\sqrt{2} = 2\tan^{-1}\frac{1}{\sqrt{2}}$$

$$= \tan^{-1}\frac{\frac{2}{\sqrt{2}}}{1 - \frac{1}{2}}$$

$$= \tan^{-1} 2\sqrt{2} = \sec^{-1} 3.$$



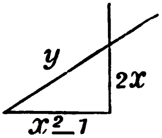
$$x^2 = 8 + 1 = 9$$
$$x = 3$$

Ex. 2. Show that $2\cot^{-1} x = \operatorname{cosec}^{-1}\frac{1+x^2}{2x}$.

$$2\cot^{-1} x = 2\tan^{-1}\frac{1}{x}$$ (xlii. 5.)

$$= \tan^{-1}\frac{\frac{2}{x}}{1 - \frac{1}{x^2}} = \tan^{-1}\frac{2x}{x^2 - 1}$$

$$= \operatorname{cosec}^{-1}\frac{1+x^2}{2x}.$$



$$y^2 = 4x^2 + x^4 - 2x^2 + 1$$
$$= x^4 + 2x^2 + 1$$
$$y = x^2 + 1.$$

Ex. 3. Show that $\tan^{-1}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\tan^{-1}\sqrt{\frac{3}{2}}=\frac{3\pi}{4}$.

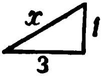
$$\text{Left-hand member}=\tan^{-1}\left\{\frac{\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\frac{\sqrt{3}}{\sqrt{2}}}{1-\frac{\sqrt{3}}{\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}}\right\} \qquad \text{(xii. 9.)}$$

$$=\tan^{-1}\left\{\frac{\sqrt{6}+2+3-\sqrt{6}}{\sqrt{6}-2-3-\sqrt{6}}\right\}$$

$$=\tan^{-1}\frac{5}{-5}=\tan^{-1}(-1)=\frac{3\pi}{4}.$$

Ex. 4. Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}=\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3$.

$\cot^{-1}3=\sin^{-1}\frac{1}{\sqrt{10}}$.



$x^2=9+1=10$

$x=\sqrt{10}$.

$$\sin^{-1}\frac{1}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{10}}=\sin^{-1}\left\{\frac{1}{\sqrt{5}}\sqrt{1-\frac{1}{10}}+\frac{1}{\sqrt{10}}\sqrt{1-\frac{1}{5}}\right\}$$

$$=\sin^{-1}\left\{\frac{3}{5\sqrt{2}}+\frac{2}{5\sqrt{2}}\right\}$$

$$=\sin^{-1}\frac{1}{\sqrt{2}}=\frac{\pi}{4}. \qquad \text{(xli. 5.)}$$

$$\text{Also }\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}=\tan^{-1}1=\frac{\pi}{4}.$$

Ex. 5. Given $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$, find θ.

$$2\tan^{-1}(\cos\theta)=\tan^{-1}\left(\frac{2\cos\theta}{1-\cos^2\theta}\right) \qquad \text{(xlv. 7.)}$$

$$\tan^{-1}\left(\frac{2\cos\theta}{\sin^2\theta}\right)=\tan^{-1}(2\operatorname{cosec}\theta)$$

$$\frac{2\cos\theta}{\sin^2\theta}=\frac{2}{\sin\theta}$$

$\sin\theta=0$, or $\sin\theta=\cos\theta$

$\therefore\ \theta=0=n\pi$, or $\tan\theta=1$

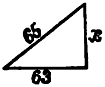
$$\therefore\ \theta=\frac{\pi}{4}=n\pi+\frac{\pi}{4}.$$

Ex. 6. Show that $\tan^{-1}(\cot A) - \tan^{-1}(\tan A) = n\pi + \frac{\pi}{2} - 2A$.

$$\tan^{-1}(\cot A) - \tan^{-1}(\tan A) = \tan^{-1}\frac{\cot A - \tan A}{1 + \cot A \tan A} = \tan^{-1}\frac{1 - \tan^2 A}{2\tan A}$$

$$= \tan^{-1}\cot 2A = \tan^{-1}\tan\left(\frac{\pi}{2} - 2A\right) = n\pi + \frac{\pi}{2} - 2A. \qquad \text{(xliii. 4.)}$$

Ex. 7. Prove that $\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}$.



$$x^2 = 65^2 - 63^2$$
$$= (65 + 63)(65 - 63)$$
$$= 256$$
$$x = 16.$$

$$\cos^{-1}\frac{63}{65} = \tan^{-1}\frac{16}{63}.$$

$$2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{2}{5}}{1 - \frac{1}{25}} = \tan^{-1}\frac{5}{12}.$$

$$\therefore \tan^{-1}\frac{16}{63} + \tan^{-1}\frac{5}{12} = \tan^{-1}\frac{\frac{16}{63} + \frac{5}{12}}{1 - \frac{16}{63}\cdot\frac{5}{12}}$$

$$= \tan^{-1}\frac{507}{676}$$

$$= \tan^{-1}\frac{3}{4} \qquad \text{(xxxiii. 5.)}$$

$$= \sin^{-1}\frac{3}{5}.$$

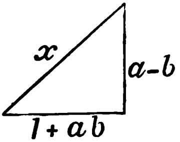
Ex. 8. Prove that $\cot^{-1}\frac{1-a}{1+a} - \cot^{-1}\frac{1-b}{1+b} = \sin^{-1}\frac{a-b}{\sqrt{1+a^2}\sqrt{1+b^2}}$.

$$\text{Left side} = \tan^{-1}\frac{1+a}{1-a} - \tan^{-1}\frac{1+b}{1-b} \qquad \text{(iii. 2.)}$$

$$= \tan^{-1}\frac{\frac{1+a}{1-a} - \frac{1+b}{1-b}}{1 + \frac{1+a}{1-a}\cdot\frac{1+b}{1-b}}$$

$$= \tan^{-1}\frac{a-b}{1+ab}$$

$$= \sin^{-1}\frac{a-b}{\sqrt{(1+a^2)(1+b^2)}}.$$



$$x^2 = (a-b)^2 + (1+ab)^2$$
$$x = 1 + a^2 + b^2 + a^2b^2$$
$$= (1+a^2)(1+b^2)$$
$$x = \sqrt{(1+a^2)(1+b^2)}$$

Ex. 9. If $u=\cot^{-1}(\cos a)^{\frac{1}{2}}-\tan^{-1}(\cos a)^{\frac{1}{2}}$, prove that $\sin u=\tan^2\frac{a}{2}$.

$$u=\tan^{-1}\frac{1}{\sqrt{\cos a}}-\tan^{-1}\sqrt{\cos a}=\tan^{-1}\frac{\frac{1}{\sqrt{\cos a}}-\sqrt{\cos a}}{1+\frac{1}{\sqrt{\cos a}}\sqrt{\cos a}}$$

$$u=\tan^{-1}\frac{1-\cos a}{2\sqrt{\cos a}}$$

$$\tan u=\frac{1-\cos a}{2\sqrt{\cos a}} \qquad \text{(lxxiv. 1.)}$$

$$\therefore \sin u=\frac{\tan u}{\sqrt{1+\tan^2 u}}=\frac{\frac{1-\cos a}{2\sqrt{\cos a}}}{\sqrt{1+\frac{(1-\cos a)^2}{4\cos a}}}=\frac{1-\cos a}{1+\cos a}=\tan^2\frac{a}{2}.$$

Ex. 10. Prove that—

$$2\tan^{-1}\left[\tan\frac{a}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right]=\tan^{-1}\left\{\frac{\sin a\cos\beta}{\sin\beta+\cos a}\right\}.$$

$$\text{Left side}=\tan^{-1}\left\{\frac{2\tan\frac{a}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{1-\tan^2\frac{a}{2}\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}\right\} \qquad \text{(xcii. 2.)}$$

$$=\tan^{-1}\left\{\frac{2\,\frac{1-\cos a}{\sin a}\cdot\frac{1-\sin\beta}{\cos\beta}}{1-\frac{(1-\sin\beta)^2}{\cos^2\beta}\cdot\frac{(1-\cos a)^2}{\sin^2 a}}\right\}$$

$$=\tan^{-1}\frac{2\sin a\cos\beta\,(1-\cos a)\,(1-\sin\beta)}{\cos^2\beta\,\sin^2 a-(1-\sin\beta)^2\,(1-\cos a)^2}$$

$$=\tan^{-1}\frac{2\sin a\cos\beta\,(1-\cos a)\,(1-\sin\beta)}{(1-\sin^2\beta)\,(1-\cos^2 a)-(1-\sin\beta)^2\,(1-\cos a)^2}$$

$$=\tan^{-1}\frac{2\sin a\cos\beta}{(1+\sin\beta)\,(1+\cos a)-(1-\sin\beta)\,(1-\cos a)}$$

$$=\tan^{-1}\frac{2\sin a\cos\beta}{2\,(\sin\beta+\cos a)}=\tan^{-1}\frac{\sin a\cos\beta}{\sin\beta+\cos a}.$$

XIII.

Logarithms to base 10: Use of Tables.

1. Def. **The Logarithm of a number to a given base is the index of the power to which the base must be raised to give the number;**

$$\text{or, number} = (\text{base})^{\log}$$

$$\text{if } \log_a m = x, \quad \text{then } a^x = m.$$

2. $\operatorname{Log}_a a = 1.$

3. $\operatorname{Log}_a 1 = 0.$

4. $\operatorname{Log}_a 0 = -\infty. \ (a > 1).$

5. $\operatorname{Log}_a mn = \log_a m + \log_a n.$

6. $\operatorname{Log}_a \frac{m}{n} = \log_a m - \log_a n.$

7. $\operatorname{Log}_a m^p = p \log_a m.$

8. $\operatorname{Log}_a m^{\frac{1}{p}} = \frac{1}{p} \log_a m.$

9. $\operatorname{Log}_a b \times \log_b a = 1.$

10. $\operatorname{Log}_a m = \log_b m \times \frac{1}{\log_b a}.$

$\frac{1}{\log_b a}$ is called the Modulus, or multiplier for transforming the log of a number to base b to the log of the same number to base a.

11. The system of logarithms, where the base is 10, is called the **Common System**, and is used in practical calculations. In theoretical investigations **Napierian Logarithms** are used, in which the base is *e*.

12. PROP. $\log_{10}(N \times 10^n) = \log_{10} N + n$,

$\log_{10}(N \div 10^n) = \log_{10} N - n$.

13. **The Mantissa is the decimal part of a logarithm, and is always kept positive.**

The Characteristic is the integral part of a logarithm, and may be either positive or negative.

14. The logs of all numbers consisting of the same digits in the same order have the same mantissa : thus, the mantissa of the logs of 1234, 123·4, ·1234, ·001234 is ·0913152 (from the Tables).

15. The characteristic of the log of a number is *one less* than the number of digits in the whole number.

When the number has no integral figures, the characteristic of the log is *negative*, and is *one more* than the number of cyphers which precede the first significant digit, that is, the number of cyphers immediately after the decimal point.

Thus the characteristics of the logs of 1234, 123·4, ·1234, 12340, ·001234, are respectively 3, 2, $\bar{1}$, 4, $\bar{3}$.

16. When a certain number of decimal places is required, strike out the rest of the figures, and increase the last retained figure by 1, if the first figure struck off is 5, or greater than 5.

EXAMPLES.

Ex. 1. What are the logarithms to base 3 of 81, $\frac{1}{81}$, and $\sqrt[4]{729}$? How many positive integers are there whose logs to base 3 have 2 for a characteristic?

$$\text{(i.) Let } \log_3 81 = x$$

$$3^x = 81 = 3^4 \qquad \text{(lxxxii. 1.)}$$

$$\therefore x = 4.$$

$$\text{(ii.) Let } \log_3 \frac{1}{81} = x$$

$$3^x = \frac{1}{81} = \frac{1}{3^4} = 3^{-4}$$

$$\therefore x = -4.$$

$$\text{(iii.) Let } \log_3 \sqrt[4]{729} = x$$

$$3^x = (729)^{\frac{1}{4}} = 3^{\frac{6}{4}} = 3^{\frac{3}{2}}$$

$$\therefore x = \frac{3}{2}.$$

$$\text{(iv.) } \log_3 9 = 2 \text{ and } \log_3 27 = 3,$$

$\therefore$ the logs of all positive integers between 9 and 26 have 2 for a characteristic, *i.e.* 18 integers.

Ex. 2. If a, b, c be any three numbers, then $\log_a b \times \log_c a = \log_c b$.

$$\text{Let } \log_a b = x, \text{ then } a^x = b$$

$$\log_c a = y, \text{ then } c^y = a \qquad \text{(lxviii. 8.)}$$

$$a^x = c^{xy}$$

$$\therefore b = c^{xy}$$

$$xy = \log_c b$$

$$\therefore \log_a b \times \log_c a = \log_c b.$$

Ex. 3. If $\log_{10} 2 = {\cdot}30103$, find the logarithms of 5, $\frac{1}{125}$, and $4\sqrt{{\cdot}005}$ to the base 10.

$$\begin{aligned}
\log 5 &= \log \frac{10}{2} = \log 10 - \log 2 \\
&= 1 - \log 2 \\
&= 1 - {\cdot}30103 = {\cdot}69897.
\end{aligned}$$

$$\begin{aligned}
\log \frac{1}{125} &= \log 1 - \log 125 = -3 \log 5 = -2{\cdot}09691 \\
&= (3 - 2{\cdot}09691) - 3 \\
&= {\cdot}90309 - 3 && \text{(xxxiv. 7.)} \\
&= \bar{3}{\cdot}90309.
\end{aligned}$$

$$\begin{aligned}
\log 4\sqrt{{\cdot}005} &= \log 4 + \frac{1}{2} \log {\cdot}005 \\
&= 2 \log 2 + \frac{1}{2}(\bar{3}{\cdot}69897) \\
&= {\cdot}60206 + \bar{2}{\cdot}84948 \\
&= \bar{1}{\cdot}45154.
\end{aligned}$$

$$\begin{aligned}
&\frac{\bar{3}{\cdot}69897}{2} \\
&= \frac{1{\cdot}69897 - 4}{2} \\
&= {\cdot}84948 - 2 \\
&= \bar{2}{\cdot}84948
\end{aligned}$$

Ex. 4. Find $\sqrt[5]{{\cdot}002}$ having given $\log 5 = {\cdot}69897$ and $\log 2{\cdot}8854 = {\cdot}460206$.

$$\begin{aligned}
x &= \sqrt[5]{{\cdot}002} \\
\log x &= \frac{1}{5} \log {\cdot}002 \\
&= \frac{1}{5}(\bar{3}{\cdot}30103) \\
&= \bar{1}{\cdot}460206 \\
&= \log {\cdot}28854 \\
\therefore\ x &= {\cdot}28854.
\end{aligned}$$

$$\begin{aligned}
\log 2 &= \log \frac{10}{5} && \text{(cv. 1.)} \\
&= 1 - \log 5 \\
&= 30103.
\end{aligned}$$

Ex. 5. Given $\log 4\cdot96 = \cdot6954817$, $\log 4\cdot9601 = \cdot6954904$, find the logarithms of 496010, ·000496, and 49600·25.

$$\log 496010 = 5\cdot6954904.$$

$$\log \cdot000496 = \bar{4}\cdot6954904. \qquad \text{(xxxiii. 6.)}$$

$$\log 4960025 = \,?$$

$$\left.\begin{array}{l}\log 4960000 = 6\cdot6954817\\ \log 4960100 = 6\cdot6954904\end{array}\right\} 87 \qquad \frac{25}{100} \text{ of } 87 = \cdot0000022$$

$$\begin{array}{r}\cdot6954817\\ \hline \cdot6954839\end{array}$$

$$\therefore \log 49600\cdot25 = 4\cdot6954839.$$

Ex. 6. Given $\log 5 = \cdot699$, how many digits are there in 2^{63}?

$$\begin{aligned} x &= 2^{63} \\ \log x &= 63 \log 2 \\ &= 63\,(1 - \log 5) \qquad \text{(cxiv. 1.)} \\ &= 63\,(\cdot301) \\ &= 18\cdot963 \end{aligned}$$

$\therefore$ x contains 19 digits.

17.—*Tabular Logarithms of the Trigonometrical Ratios.*

The Trig-ratios, being real numerical quantities, have corresponding logarithms.

Since the sines and cosines of all angles, and the tangents of angles less than 45° are less than unity, the log-sines and the log-cosines of all angles, and the log-tangents of angles less than 45° will have negative characteristics. To avoid registering negative characteristics, the tabular logs of the trig-ratios are increased by 10.

For example: $\sin 30° = \frac{1}{2} = \cdot5.$

$$\therefore \log \sin 30° = \log \cdot5 = \bar{1}\cdot6989700$$

and in the tables we find $\text{L} \sin 30 = 9\cdot6989700$.

In using the tables it must be carefully remembered that as the angle increases the L sine, L sec, L tan increase, whereas the L cos, L cot, L cosec decrease.

$$\text{Also } \tan A = \frac{\sin A}{\cos A}$$

$$\text{L} \tan A - 10 = \text{L} \sin A - 10 - (\text{L} \cos A - 10)$$

$$\therefore\ \text{L} \tan A = 10 + \text{L} \sin A - \text{L} \cos A.$$

$$\sec A = \frac{1}{\cos A}$$

$$\text{L} \sec A - 10 = -(\text{L} \cos A - 10)$$

$$\text{L} \sec A = 20 - \text{L} \cos A.$$

Examples.

Ex. 1. Given L sin 73° 20′ 10″ = 9·9813671,

L sin 73° 20′ 20″ = 9·9813734,

find L sin 73° 20′ 15·7″.

$$\text{L} \sin 73^\circ\ 20'\ 15\cdot7'' = ?$$

$$\left.\begin{array}{l} \text{L} \sin 73^\circ\ 20'\ 10'' = 9\cdot9813671 \\ \text{L} \sin 73^\circ\ 20'\ 20'' = 9\cdot9813734 \end{array}\right\} 63 \qquad \text{(xxxix. 11.)}$$

$$\begin{array}{rrr} \frac{5\cdot7}{10} \text{ of } 63 = & \cdot57 \times 63 = & \cdot0000036 \\ & \cdot57 & 9\cdot9813671 \\ \cline{2-3} & 441 & 9\cdot9813707 \\ & 315 & \\ \cline{2-2} & \cdot000003591 & \end{array}$$

L sin 73° 20′ 15·7″ = 9·9813707.

Ex. 2. Given $\text{L}\cos 55^\circ\ 11' = 9\cdot7565999$, $\text{L}\cos 55^\circ\ 12' = 9\cdot7564182$, find the tabular logarithm of the cosine of $55^\circ\ 11'\ 12''$.

$$\text{L}\cos 55^\circ\ 11'\ 12'' = \text{?}$$

$$\left.\begin{aligned}\text{L}\cos 55^\circ\ 11' &= 9\cdot7565999\\ \text{L}\cos 55^\circ\ 12' &= 9\cdot7564182\end{aligned}\right\}1817$$

$$\frac{12}{60}\text{ of }1817 = \frac{1}{5}\text{ of }1817 = 363$$

$$\begin{array}{r} 9\cdot7565999 \\ \cdot0000363 \\ \hline \end{array}\qquad\qquad \text{(xli. 10.)}$$

$$\text{L}\cos 55^\circ\ 11'\ 12'' = 9\cdot7565636$$

Ex. 3. Given $\text{L}\tan 38^\circ\ 16' = 9\cdot8969714$, determine $\text{L}\cot 38^\circ\ 16'$; and given $\text{L}\tan 38^\circ\ 16'\ 10'' = 9\cdot8970147$, $\text{L}\tan 38^\circ\ 16'\ 20'' = 9\cdot8970580$, find the angle whose logarithmic tangent is $9\cdot8970365$.

$$\text{(i.)}\quad \cot 38^\circ\ 16' = \frac{1}{\tan 38^\circ\ 16'}$$

$$\begin{aligned} \text{L}\cot 38^\circ\ 16' - 10 &= -(\text{L}\tan 38^\circ\ 16' - 10)\\ \therefore\ \text{L}\cot 38^\circ\ 16'' &= 20 - \text{L}\tan 38^\circ\ 16'\\ &= 20 - 9\cdot8969714 \qquad\qquad \text{(xxxviii. 11.)}\\ &= 10\cdot1030286. \end{aligned}$$

$$\begin{aligned} \text{(ii.)}\quad \text{L}\tan a &= 9\cdot8970365\\ \text{L}\tan 38^\circ\ 16'\ 10'' &= 9\cdot8970147\\ \text{L}\tan 38^\circ\ 16'\ 20'' &= 9\cdot8970580 \end{aligned}\ \begin{array}{l}\left.\begin{matrix}\\ \\ \end{matrix}\right\}218\\ \quad\left.\begin{matrix}\\ \\ \end{matrix}\right\}433\end{array}\qquad \frac{218}{433}\times 10'' = 5''$$

$$\therefore\ a = 38^\circ\ 16'\ 15''.$$

XIV.

Relations between the Sides of a Triangle, and the Trig-Ratios of the Angles.

a, b, c are the sides opposite to the angles A, B, C respectively.

s = half the perimeter of the triangle $=\frac{1}{2}(a+b+c)$.

1. Prop. The sines of the angles are proportional to the opposite sides.

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}.$$

2. Prop. Each side is equal to the sum of the product of each of the other sides into the cosine of the angle it makes with the first side.

$$a=b\cos C+c\cos B,\quad b=a\cos C+c\cos A,\quad c=a\cos B+b\cos A.$$

$$\therefore\ a+b+c=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C. \qquad \text{(vii. 5.)}$$

3. Prop. Express the cosine of an angle in terms of the sides.

$$b^2=a^2+c^2-2ac\cos B,\quad a^2=b^2+c^2-2bc\cos A,\quad c^2=a^2+b^2-2ab\cos C.$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc},\qquad \cos B=\frac{a^2+c^2-b^2}{2ac},\qquad \cos C=\frac{a^2+b^2-c^2}{2ab}.$$

4. Prop. Express the Trig-ratios of the half angle in terms of the sides.

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},\ \sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}},\ \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}.$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},\qquad \cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}},\qquad \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}.$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\ \tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}},\ \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Since any angle of a triangle is < 180°, the half angle is < 90°, and all the trig-ratios are +; there is therefore no ambiguity of sign.

5. **Prop.** Express the sine of an angle in terms of the sides.

$$\sin A=\frac{2S}{bc},\quad \sin B=\frac{2S}{ac},\quad \sin C=\frac{2S}{ab},$$

$$\text{where } S=\sqrt{s(s-a)(s-b)(s-c)}.$$

6. **Prop.** $\operatorname{Tan}\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$

7. **Prop.** $\operatorname{Sin}\frac{A-B}{2}=\frac{a-b}{c}\cos\frac{C}{2}.$

8. **Prop.** $\operatorname{Cos}\frac{A-B}{2}=\frac{a+b}{c}\sin\frac{C}{2}.$

9. **Prop.** Given $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$, deduce

$$a=b\cos C+c\cos B.$$

Let $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$

$$\left.\begin{aligned}a&=k\sin A\\ b&=k\sin B\\ c&=k\sin C\end{aligned}\right\}$$

$$b\cos C=k\sin B\cos C,\qquad c\cos B=k\sin C\cos B$$

$$\begin{aligned}\therefore\ b\cos C+c\cos B&=k(\sin B\cos C+\cos B\sin C)\\ &=k\sin(B+C)\\ &=k\sin A\\ &=a.\end{aligned}$$

10. **Prop.** Given $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$, deduce

$$a^2=b^2+c^2-2bc\cos A.$$

Let $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$

$$\left.\begin{aligned}a&=k\sin A\\ b&=k\sin B\\ c&=k\sin C\end{aligned}\right\}$$

$$\begin{aligned}\therefore\ b^2+c^2-2bc\cos A&=k^2\sin^2 B+k^2\sin^2 C-2k^2\sin B\sin C\cos A\\ &=k^2\{\sin^2 B+\sin^2 C-2\sin B\sin C\cos A\}\\ &=k^2\{\sin^2 B+\sin C(\sin C-2\sin B\cos A)\}\\ &=k^2\{\sin^2 B+\sin(A+B)(\sin\overline{A+B}-2\sin B\cos A)\}\\ &=k^2\{\sin^2 B+\sin(A+B)\sin(A-B)\}\\ &=k^2\{\sin^2 B+\sin^2 A-\sin^2 B\}\\ &=k^2\sin^2 A\\ &=a^2.\end{aligned}$$

11. **Prop.** Given $a=b\cos C+c\cos B$, $b=a\cos C+c\cos A$, $c=a\cos B+b\cos A$, deduce $a^2=b^2+c^2-2bc\cos A$.

Multiply the equations by a, b, c respectively,

$$\left.\begin{aligned}a^2&=ab\cos C+ac\cos B\\ b^2&=ab\cos C+bc\cos A\\ c^2&=ac\cos B+bc\cos A\end{aligned}\right\}\begin{aligned}&(1)\\&(2)\\&(3)\end{aligned}\qquad\text{(xlix. 7.)}$$

add (2) and (3) and subtract (1)

$$b^2+c^2-a^2=2bc\cos A$$

$$\therefore\ a^2=b^2+c^2-2bc\cos A.$$

12. **Prop.** Prove $\tan\dfrac{A-B}{2}=\dfrac{a-b}{a+b}\cot\dfrac{C}{2}$.

$$\text{Let } \frac{\sin A}{a}=\frac{\sin B}{b}=k$$

$$\sin A=ak$$

$$\sin B=bk$$

$$\sin A-\sin B=(a-b)k$$

$$\sin A+\sin B=(a+b)k$$

$$\frac{\sin A-\sin B}{\sin A+\sin B}=\frac{a-b}{a+b}$$

$$\frac{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}{2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}}=\frac{a-b}{a+b}$$

$$\tan\frac{A-B}{2}\cot\frac{A+B}{2}=\frac{a-b}{a+b}$$

$$\tan\frac{A-B}{2}\tan\frac{C}{2}=\frac{a-b}{a+b}, \quad \because \frac{A+B}{2}=90^\circ-\frac{C}{2}$$

$$\therefore \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

13. **Prop.** Prove $\sin\dfrac{A-B}{2}=\dfrac{a-b}{c}\cos\dfrac{C}{2}$.

$$\text{Let } \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

$$\left.\begin{aligned}a&=k\sin A\\ b&=k\sin B\\ c&=k\sin C\end{aligned}\right\}$$

$$\therefore \frac{a-b}{c}=\frac{k(\sin A-\sin B)}{k\sin C}=\frac{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}{2\sin\dfrac{C}{2}\cos\dfrac{C}{2}}$$

$$=\frac{\sin\dfrac{A-B}{2}}{\cos\dfrac{C}{2}}, \quad \text{since } \cos\frac{A+B}{2}=\sin\frac{C}{2}.$$

10. **Prop.** Given $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$, deduce

$$a^2=b^2+c^2-2bc\cos A.$$

Let $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$

$$\left.\begin{array}{l} a=k\sin A \\ b=k\sin B \\ c=k\sin C \end{array}\right\}$$

$$\begin{aligned} \therefore\ b^2+c^2-2bc\cos A &= k^2\sin^2 B+k^2\sin^2 C-2k^2\sin B\sin C\cos A \\ &= k^2\{\sin^2 B+\sin^2 C-2\sin B\sin C\cos A\} \\ &= k^2\{\sin^2 B+\sin C(\sin C-2\sin B\cos A)\} \\ &= k^2\{\sin^2 B+\sin(A+B)(\sin\overline{A+B}-2\sin B\cos A)\} \\ &= k^2\{\sin^2 B+\sin(A+B)\sin(A-B)\} \\ &= k^2\{\sin^2 B+\sin^2 A-\sin^2 B\} \\ &= k^2\sin^2 A \\ &= a^2. \end{aligned}$$

11. **Prop.** Given $a=b\cos C+c\cos B$, $b=a\cos C+c\cos A$, $c=a\cos B+b\cos A$, deduce $a^2=b^2+c^2-2bc\cos A$.

Multiply the equations by a, b, c respectively,

$$\left.\begin{array}{l} a^2=ab\cos C+ac\cos B \\ b^2=ab\cos C+bc\cos A \\ c^2=ac\cos B+bc\cos A \end{array}\right\}\begin{array}{l}(1)\\(2)\\(3)\end{array} \qquad \text{(xlix. 7.)}$$

add (2) and (3) and subtract (1)

$b^2+c^2-a^2=2bc\cos A$

$$\therefore\ a^2=b^2+c^2-2bc\cos A.$$

12. **Prop.** Prove $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$.

$$\text{Let } \frac{\sin A}{a}=\frac{\sin B}{b}=k$$

$$\sin A=ak$$

$$\sin B=bk$$

$$\sin A-\sin B=(a-b)k$$

$$\sin A+\sin B=(a+b)k$$

$$\frac{\sin A-\sin B}{\sin A+\sin B}=\frac{a-b}{a+b}$$

$$\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}=\frac{a-b}{a+b}$$

$$\tan\frac{A-B}{2}\cot\frac{A+B}{2}=\frac{a-b}{a+b}$$

$$\tan\frac{A-B}{2}\tan\frac{C}{2}=\frac{a-b}{a+b}, \quad \because \frac{A+B}{2}=90^\circ-\frac{C}{2}$$

$$\therefore \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

13. **Prop.** Prove $\sin\frac{A-B}{2}=\frac{a-b}{c}\cos\frac{C}{2}$.

$$\text{Let } \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

$$\left.\begin{aligned} a&=k\sin A\\ b&=k\sin B\\ c&=k\sin C\end{aligned}\right\}$$

$$\therefore \frac{a-b}{c}=\frac{k(\sin A-\sin B)}{k\sin C}=\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$=\frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}, \quad \text{since } \cos\frac{A+B}{2}=\sin\frac{C}{2}.$$

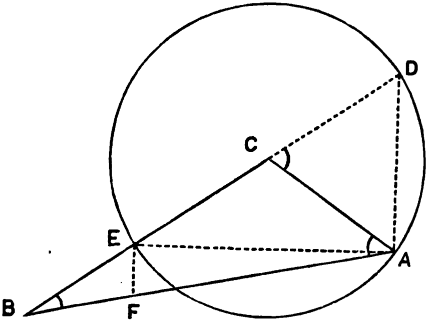
14. Prop. Prove $\cos\frac{A-B}{2}=\frac{a+b}{c}\sin\frac{C}{2}$.

$$\frac{a+b}{c}=\frac{k(\sin A+\sin B)}{k\sin C}=\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$=\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}},\quad \text{since } \sin\frac{A+B}{2}=\cos\frac{C}{2}.$$

15. Prop. The three previous formulæ proved Geometrically.

(i.) Let ABC be the given triangle, where $a > b$.



Describe a circle with centre C and radius b, cutting BC in E.

Produce BC to D, join AD, AE, and draw EF parallel to AD; it is therefore perpendicular to AE, since DAE in a semi-circle is a right angle.

Exterior angle $ACD=A+B$

$$=2AED \text{ at circumference.}$$

$$\therefore\ AED=\frac{A+B}{2},$$

interior angle $EAB=AED-B=\frac{A+B}{2}-B=\frac{A-B}{2}$.

$$\therefore\ \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}=\frac{\frac{EF}{EA}}{\frac{DA}{EA}}=\frac{EF}{DA}=\frac{BE}{BD}, \text{ by similar triangles } BEF, BDA$$

$$=\frac{BC-CE}{BC+CD}=\frac{a-b}{a+b}.$$

(ii.) Since $BAD = 90^\circ + \frac{A-B}{2}$, and $BDA = \frac{C}{2}$.

$$\frac{BD}{BA} = \frac{\sin BAD}{\sin BDA} = \frac{\sin\left(90^\circ + \frac{A-B}{2}\right)}{\sin\frac{C}{2}}$$

$$\therefore \frac{a+b}{c} = \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}.$$

(iii.) Since $\frac{BE}{BA} = \frac{\sin BAE}{\sin BEA} = \frac{\sin\frac{A-B}{2}}{\sin\left(180^\circ - \frac{A+B}{2}\right)}$

$$\therefore \frac{a-b}{c} = \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}} = \frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}.$$

16. **Prop.** Prove that $\tan\left(\frac{A}{2}+B\right) = \frac{c+b}{c-b}\tan\frac{A}{2}$.

It has been proved that

$$\tan\frac{C-B}{2} = \frac{c-b}{c+b}\cot\frac{A}{2}$$

Invert, $\cot\frac{C-B}{2} = \frac{c+b}{c-b}\tan\frac{A}{2}$

$$\tan\left(90^\circ - \frac{C-B}{2}\right) = \frac{c+b}{c-b}\tan\frac{A}{2}$$

$$\tan\left(\frac{A+B+C}{2} - \frac{C-B}{2}\right) = \frac{c+b}{c-b}\tan\frac{A}{2}$$

$$\therefore \tan\left(\frac{A}{2}+B\right) = \frac{c+b}{c-b}\tan\frac{A}{2}.$$

EXAMPLES.

Ex. 1. If the tangents of the half angles of a triangle are in H.P., show that the sides of the triangle are in A.P.

By Question, $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in H.P. (lxxxviii. 3.)

$$\cot\frac{A}{2},\quad \cot\frac{B}{2},\quad \cot\frac{C}{2} \text{ are in A.P.}$$

$$\sqrt{\frac{s(s-a)}{(s-b)(s-c)}},\quad \sqrt{\frac{s(s-b)}{(s-a)(s-c)}},\quad \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \text{ are in A.P.}$$

$$s-a,\quad s-b,\quad s-c \text{ are in A.P.}$$

$$\therefore\ a,\quad b,\quad c \quad ,, \quad ,,$$

Ex. 2. In the triangle ABC, if $\sin A$, $\sin B$, $\sin C$ are in H.P., then $1-\cos A$, $1-\cos B$, $1-\cos C$ are also in H.P.

If $1-\cos A$, $1-\cos B$, $1-\cos C$ are in H.P.

$$2\sin^2\frac{A}{2},\quad 2\sin^2\frac{B}{2},\quad 2\sin^2\frac{C}{2} \quad ,, \quad ,,$$

$$\frac{2(s-b)(s-c)}{bc},\quad \frac{2(s-a)(s-c)}{ac},\quad \frac{2(s-a)(s-b)}{ab} \text{ are in H.P.}$$

$$\frac{bc}{2(s-b)(s-c)},\quad \frac{ac}{2(s-a)(s-c)},\quad \frac{ab}{2(s-a)(s-b)} \text{ are in A.P.}$$

$$\frac{s-a}{a},\quad \frac{s-b}{b},\quad \frac{s-c}{c} \text{ are in A.P.}$$

$$\frac{s}{a},\quad \frac{s}{b},\quad \frac{s}{c} \text{ are in A.P.}$$

$$\frac{1}{a},\quad \frac{1}{b},\quad \frac{1}{c} \quad ,, \quad ,,$$

$$\frac{1}{\sin A},\quad \frac{1}{\sin B},\quad \frac{1}{\sin C} \text{ are in A.P.}$$

$$\sin A,\quad \sin B,\quad \sin C \quad ,, \quad \text{H.P.}$$

Ex. 3. If in the triangle ABC the angle A be three times the angle B, prove that $\sin B = \frac{1}{2}\sqrt{\frac{3b-a}{b}}$.

$$A = 3B, \quad \therefore \sin A = \sin 3B = 3\sin B - 4\sin^3 B.$$

$$\frac{\sin A}{\sin B} = 3 - 4\sin^2 B$$

$$\frac{a}{b} = 3 - 4\sin^2 B \qquad \text{(lxxxvi. 3.)}$$

$$4\sin^2 B = \frac{3b-a}{b}$$

$$\therefore \sin B = \frac{1}{2}\sqrt{\frac{3b-a}{b}}.$$

Ex. 4. If a, b, c, the sides of a triangle, be in A.P., prove that $2\sin\frac{A}{2}\sin\frac{C}{2} = \sin\frac{B}{2}$.

Since a, b, c, are in A.P., (lxxx. 2.)

$$2b = a + c$$

$$2\sin\frac{A}{2}\sin\frac{C}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc} \times \frac{(s-a)(s-b)}{ab}}$$

$$= \frac{2(s-b)}{b}\sqrt{\frac{(s-a)(s-c)}{ac}}$$

$$= \frac{a+c-b}{b}\sin\frac{B}{2}$$

$$= \frac{2b-b}{b}\sin\frac{B}{2} = \sin\frac{B}{2}.$$

Ex. 5. In any triangle, prove that $\dfrac{a^2+b^2-ab\cos C}{a\sin A+b\sin B+c\sin C}=\dfrac{a}{2\sin A}$.

$$\text{Let } \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

$$\begin{aligned} a&=k\sin A,\ \therefore a^2=ka\sin A\\ b&=k\sin B,\ \therefore b^2=kb\sin B \qquad \text{(ii. 5.)}\\ c&=k\sin C,\ \therefore c^2=kc\sin C\end{aligned}$$

$$a^2+b^2+c^2=k\,(a\sin A+b\sin B+c\sin C)$$

$$k=\frac{a^2+b^2+c^2}{a\sin A+b\sin B+c\sin C}$$

$$\frac{a}{\sin A}=\frac{a^2+b^2+(a^2+b^2-2ab\cos C)}{a\sin A+b\sin B+c\sin C}$$

$$=\frac{2\,(a^2+b^2-ab\cos C)}{a\sin A+b\sin B+c\sin C}.$$

Ex. 6. In any triangle, prove that $\dfrac{a\cos B-b\cos A}{c}=\dfrac{\sin^2 A-\sin^2 B}{\sin^2 C}$.

$$\text{Left-hand member}=\frac{1}{c}\left(a\,\frac{a^2+c^2-b^2}{2ac}-b\,\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\frac{2\,(a^2-b^2)}{2c^2}$$

$$=\frac{\sin^2 A-\sin^2 B}{\sin^2 C}. \qquad \text{(xliii. 7.)}$$

Ex. 7. Prove that $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\dfrac{a^2+b^2}{a^2+c^2}$.

$$\text{Left-hand member}=\frac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)} \qquad \text{(xxxiv. 8.)}$$

$$=\frac{1-(\cos^2 A-\sin^2 B)}{1-(\cos^2 A-\sin^2 C)}$$

$$=\frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C}$$

$$=\frac{a^2+b^2}{a^2+c^2}.$$

Ex. 8. If α, β, γ be the perpendiculars from the angles of the triangle ABC upon the opposite sides a, b, c, prove that—

$$a \sin A + b \sin B + c \sin C = 2\,(\alpha \cos A + \beta \cos B + \gamma \cos C).$$

$$\alpha = b \sin C = c \sin B, \quad \beta = a \sin C = c \sin A, \quad \gamma = a \sin B = b \sin A$$

$$\therefore\ 2\alpha = b \sin C + c \sin B,\ 2\beta = a \sin C + c \sin A,\ 2\gamma = a \sin B + b \sin A$$

Right-hand member—

$$= \cos A\,(b \sin C + c \sin B) + \cos B\,(a \sin C + c \sin A)$$
$$+ \cos C\,(a \sin B + b \sin A)$$
$$= a \sin (B + C) + b \sin (A + C) + c \sin (A + B) \qquad \text{(xlviii. 8.)}$$
$$= a \sin A + b \sin B + c \sin C.$$

Ex. 9. Prove that $(a+b+c)\left(\tan \frac{A}{2} + \tan \frac{B}{2}\right) = 2c \cot \frac{C}{2}$.

$$\text{Left-hand member} = 2s \frac{\sin \frac{A+B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} \qquad \text{(xxxi. 8.)}$$

$$= 2s \frac{\cos \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}}$$

$$= 2s \sqrt{\frac{\frac{s\,(s-c)}{ab}}{\frac{s\,(s-a)}{bc} \cdot \frac{s\,(s-b)}{ac}}} = 2sc \sqrt{\frac{s-c}{s(s-a)(s-b)}}$$

$$= 2c \sqrt{\frac{s(s-c)}{(s-a)\,(s-b)}} = 2c \cot \frac{C}{2}.$$

XV.

Solution of Triangles.

In every triangle there are six elements, the three sides and the three angles; and in order to solve a triangle, three elements must be given, one at least of which must be a side.

1. Given three sides a, b, c. (Eucl. i. 8.)

Quote the formula for the trig-ratio of the half angle in terms of the sides. If one angle only is to be found, the formulæ for $\sin\frac{A}{2}$ or $\cos\frac{A}{2}$ may be used; if all the angles are to be found, the most convenient formulæ are those for $\tan\frac{A}{2}$ and $\tan\frac{B}{2}$, in which the logs of the same quantities s, $s-a$, $s-b$, $s-c$ are required, whereas the formulæ for the sine and cosine require in addition the logs of a, b, c.

Having found A and B, then $C = 180^\circ - (A+B)$.

Without the use of logarithms, the angles may be found by the formulæ $\cos A = \frac{b^2+c^2-a^2}{2bc}$, etc.

EXAMPLES.

Ex. 1. The sides a, b, c of a triangle are as $4:5:6$. Find the angle B; given $\mathrm{L}\cos 27^\circ 53' = 9{\cdot}9464040$, $\mathrm{L}\cos 27^\circ 54' = 9{\cdot}9463371$, $\log 2 = {\cdot}3010300$.

$$\left.\begin{aligned} a&=4 \\ b&=5 \\ c&=6 \end{aligned}\right\} s = \frac{15}{2} = 7{\cdot}5 \qquad \text{(viii. 5.)}$$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} = \sqrt{\frac{7{\cdot}5 \times 2{\cdot}5}{24}} = \frac{5}{\sqrt{32}} = \frac{5}{\sqrt{2^5}}$$

$$\mathrm{L}\cos\frac{B}{2} - 10 = \log 5 - \frac{5}{2}\log 2$$

$$\mathrm{L}\cos\frac{B}{2} = 10 + (1 - \log 2) - \frac{5}{2}\log 2 \quad = 11 - \frac{7}{2}\log 2 = 11 - 1{\cdot}0536050$$

$$\begin{array}{l} \mathrm{L}\cos\frac{B}{2} = 9{\cdot}9463950 \\ \mathrm{L}\cos 27^\circ 53' = 9{\cdot}9464040 \\ \mathrm{L}\cos 27^\circ 54' = 9{\cdot}9463371 \end{array} \begin{array}{l} \left.\begin{array}{l} \\ \end{array}\right\} 90 \\ \left.\begin{array}{l} \\ \end{array}\right\} 669 \end{array} \qquad \frac{90}{669} \times 60'' = 8'',$$

$$\frac{B}{2} = 27^\circ\ 53'\ 8'', \qquad \therefore\ B = 55^\circ\ 46'\ 16''.$$

Ex. 2. Given $a=3$, $b=2{\cdot}75$, $c=1{\cdot}75$, find B; given $\log 2={\cdot}3010300$, $\text{L}\tan 32^\circ\ 19'=9{\cdot}8008365$, diff. for $1'=2796$.

$$\left.\begin{array}{l} a=3\\ b=2{\cdot}75\\ c=1{\cdot}75\end{array}\right\}\ s=3{\cdot}75 \qquad\qquad \text{(iv. 6.)}$$

$$\tan\frac{B}{2}=\sqrt{\frac{(s-a)\,(s-c)}{s\,(s-b)}}=\sqrt{\frac{{\cdot}75\times2}{3{\cdot}75\times1}}=\sqrt{\frac{2}{5}}=\sqrt{\frac{4}{10}}$$

$$\text{L}\tan\frac{B}{2}-10=\frac{1}{2}\,(2\log 2-1)$$

$$\text{L}\tan\frac{B}{2}=10-\frac{1}{2}\,({\cdot}3979400)$$

$$=10-{\cdot}1989700$$

$$\left.\begin{array}{r} =9{\cdot}8010300\\ \text{L}\tan 32^\circ\ 19'=9{\cdot}8008365\end{array}\right\}\ 1935 \qquad\qquad \frac{1935}{2796}\times60''=41''$$

$$\frac{B}{2}=32^\circ\ 19'\ 41''$$

$$\therefore\ B=64^\circ\ 39'\ 22''.$$

Ex. 3. Solve the triangle, in which $a=18$, $b=12$, $c=10$, having given
$\tan 54^\circ\ 44'=1{\cdot}4140943$, $\tan 15^\circ\ 47'={\cdot}2826573$
$\tan 54^\circ\ 45'=1{\cdot}4149673$, $\tan 15^\circ\ 48'={\cdot}2829715$.

$$\left.\begin{array}{l} a=18\\ b=12\\ c=10\end{array}\right\}\ s=20 \qquad\qquad \text{(xvi. 6.)}$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}}=\sqrt{\frac{8\times10}{20\times\ 2}}=\sqrt{2}=1{\cdot}4142136$$

$$\begin{array}{rl} \tan\frac{A}{2}=1{\cdot}4142136 & \Big\}\ 1193\\ \tan 54^\circ\ 44'=1{\cdot}4140943 & \Big\}\ 8730\\ \tan 54^\circ\ 45'=1{\cdot}4149673 & \end{array} \qquad\qquad \frac{1193}{8730}\times60''=8''$$

$$\frac{A}{2}=54^\circ\ 44'\ 8'',\quad \therefore\ A=109^\circ\ 28'\ 16''.$$

$$\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\sqrt{[illegible]}=\frac{\sqrt{2}}{5}=\frac{1\cdot4142136}{5}$$

$$=\cdot2828427$$

$\tan 15°\ 47'=\cdot28[illegible]$ [illegible] [illegible] $=35''$

$\tan 15°\ 48'=\cdot28[illegible]$

$$\frac{C}{2}=15°\ 47'\ [illegible]''$$

$$C=31°\ [illegible]'\ [illegible]''$$

Ex. 4. If $a=\sqrt{6}$, $b=2$, $c=\sqrt{3}+1$, solve the triangle.

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{4+(4+2\sqrt{3})-6}{4(\sqrt{3}+1)}\qquad\text{(xcviii. 4.)}$$

$$=\frac{2(\sqrt{3}+1)}{4(\sqrt{3}+1)}=\frac{1}{2},\qquad\therefore A=60°.$$

$$\cos B=\frac{c^2+a^2-b^2}{2ca}$$

$$=\frac{(4+2\sqrt{3})+6-4}{2\sqrt{6}(\sqrt{3}+1)}$$

$$=\frac{2\sqrt{3}(\sqrt{3}+1)}{2\sqrt{6}(\sqrt{3}+1)}=\frac{1}{\sqrt{2}},\qquad\therefore B=45°.$$

$$C=180°-(A+B)=75°.$$

[illegible] side and two angles, a, B, C. (Eucl. i. 26.)

$[illegible]=180°-(B+C).$

$$[illegible]=\frac{a\sin B}{\sin A},\qquad c=\frac{a\sin C}{\sin A}$$

$[illegible]=\log a+\mathrm{L}\sin B-\mathrm{L}\sin A.$

$[illegible]\log a+\mathrm{L}\sin C-\mathrm{L}\sin A.$

EXAMPLES.

Ex. 1. In the triangle ABC, $A = 60^\circ\ 15'$, $B = 54^\circ\ 30'$, and $c = 100$, find b; given $\text{L} \sin 54^\circ\ 30' = 9{\cdot}9106860$, $\text{L} \sin 65^\circ\ 15' = 9{\cdot}9581543$, $\log {\cdot}89646 = \bar{1}{\cdot}9525317$.

$$C = 180^\circ - (114^\circ\ 45') = 65^\circ\ 15'. \qquad \text{(xxiii. 8.)}$$

$$b = \frac{c \sin B}{\sin C} = \frac{100 \sin 54^\circ\ 30'}{\sin 65^\circ\ 15'}$$

$$\begin{aligned} \log b &= \log 100 + \text{L} \sin 54^\circ\ 30' - \text{L} \sin 65^\circ\ 15' \\ &= 2 + 9{\cdot}9106860 - 9{\cdot}9581543 \\ &= 1{\cdot}9525317 \end{aligned}$$

$$\therefore\ b = 89{\cdot}646.$$

Ex. 2. If $A = 55^\circ$, $B = 65^\circ$ and $c = 270$, find a, given $\text{L} \sin 55^\circ = 9{\cdot}9133645$ $\log 3 = {\cdot}4771213$, $\log 2 = {\cdot}3010300$, $\log 2{\cdot}5553 = {\cdot}4070508$, $\log 2{\cdot}5554 = {\cdot}4072209$.

$$C = 180^\circ - 120^\circ = 60^\circ$$

$$a = \frac{c \sin A}{\sin C} = \frac{270 \sin 55^\circ}{\frac{\sqrt{3}}{2}} = \frac{540 \sin 55^\circ}{\sqrt{3}} \qquad \text{(xxiv. 8.)}$$

$$\begin{aligned} \log a &= 3 \log 3 + \log 2 + 1 + \text{L} \sin 55^\circ - 10 - \frac{1}{2} \log 3 \\ &= \frac{5}{2} \log 3 + \log 2 + \text{L} \sin 55^\circ - 9 \\ &= 1{\cdot}1928032 + {\cdot}3010300 + 9{\cdot}9133645 - 9 \end{aligned}$$

$$\begin{array}{rl} \log a = 2{\cdot}4071977 & \left.\right\} 1469 \\ \log 25553 = 4{\cdot}4070508 & \left.\right\} 1701 \\ \log 25554 = 4{\cdot}4072209 & \end{array} \qquad \frac{1469}{1701} = {\cdot}863\ \ldots$$

$$\therefore\ a = 255{\cdot}53863.$$

3. Given two sides and the included angle, a, b, C. (Eucl. i. 4.)

Substitute in the formula—

$\tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$, reduce the fraction to its lowest terms, and take logarithms,

Hence $\dfrac{A-B}{2}$ is known, and $\dfrac{A+B}{2} = 90^\circ - \dfrac{C}{2}$,

$\therefore$ A and B are found.

$$\text{Then } c = \frac{b \sin C}{\sin B}, \text{ or } \frac{a \sin C}{\sin A}.$$

$$\text{Also, } c = \frac{(a+b)\sin\dfrac{C}{2}}{\cos\dfrac{A-B}{2}}, \text{ or } \frac{(a-b)\cos\dfrac{C}{2}}{\sin\dfrac{A-B}{2}},$$

in which we require only two additional logarithms, those of $\sin\frac{C}{2}$ and $\cos\frac{A-B}{2}$, or of $\cos\frac{C}{2}$ and $\sin\frac{A-B}{2}$.

Again, c may be found without finding the adjacent angles, by means of $c=(a+b)\sin\frac{C}{2}\sec\phi$, where $\tan\phi=\frac{a-b}{a+b}\cot\frac{C}{2}$. [*See* XIX. 2 (iii.).]

EXAMPLES.

Ex. 1. If $a=5$, $b=3$, $C=60°\ 30'$, find the other angles of the triangle; given $\log 2 = \cdot 30103$, $L \cot 30°\ 15' = 10\cdot 23420$, $L \tan 23°\ 13' = 9\cdot 63240$, diff. for $1' = \cdot 00035$.

$$\begin{aligned} \tan\frac{A-B}{2} &= \frac{a-b}{a+b}\cot\frac{C}{2} \\ &= \frac{2}{8}\cot 30°\ 15' \\ &= \frac{1}{4}\cot 30°\ 15' \end{aligned}$$

$$\begin{aligned} L \tan\frac{A-B}{2} &= L \cot 30°\ 15' - 2\log 2 \\ &= 10\cdot 23420 - \cdot 60206 \end{aligned}$$

$$\left.\begin{aligned} &= 9\cdot 63214 \\ L \tan 23°\ 13' &= 9\cdot 63240 \end{aligned}\right\} 26 \qquad\qquad \text{(ix. 5.)}$$

$$\frac{26}{35}\times 60'' = \frac{312}{7} = 45''$$

$$\frac{A-B}{2} = 23°\ 13'\ 45''$$

$$\frac{A+B}{2} = 90° - 30°\ 15' = 59°\ 45'$$

$$\begin{aligned} \therefore\ A &= 82°\ 58'\ 45''. \\ B &= 36°\ 31'\ 15''. \end{aligned}$$

Ex. 2. In any triangle given that $\tan\left(\frac{A}{2}+B\right)=\frac{c+b}{c-b}\tan\frac{A}{2}$, if $3c=7b$, and $A=6° \, 37' \, 24''$, find the other angles; given

$\text{L}\tan 3° \, 18' \, 42''=8{\cdot}7624069$, $\text{L}\tan 8° \, 13' \, 50''=9{\cdot}1603083$,

$\log 2={\cdot}3010300$, diff. for $10''=1486$.

$$\tan\left(\frac{A}{2}+B\right)=\frac{c+b}{c-b}\tan\frac{A}{2}$$

$$=\frac{10}{4}\tan 3° \, 18' \, 42'' \qquad \text{(vi. 5.)}$$

$$\text{L}\tan\left(\frac{A}{2}+B\right)=1-2\log 2+\text{L}\tan 3° \, 18' \, 42''$$

$$\left.\begin{aligned}&=9{\cdot}1603469\\ \text{L}\tan 8° \, 13' \, 50''&=9{\cdot}1603083\end{aligned}\right\}386$$

$$\frac{386}{1486}\times 10''=2{\cdot}5''$$

$$\frac{A}{2}+B=8° \, 13' \, 53''$$

$$B=8° \, 13' \, 53''-3° \, 18' \, 42''$$

$$\therefore\ B=4° \, 55' \, 11''.$$

$$C=180°-11° \, 32' \, 45''=168° \, 27' \, 15''.$$

Ex. 3. If $b:c::11:10$, and $A=35° \, 25'$, apply the formula $\tan\frac{B-C}{2}=\tan^2\frac{\phi}{2}\cot\frac{A}{2}$ to find B and C, given $\log 1{\cdot}1={\cdot}041393$,

$\text{L}\cos 24° \, 37' \, 12''=9{\cdot}958607$, $\text{L}\tan 12° \, 18' \, 36''=9{\cdot}338891$,

$\text{L}\cot 17° \, 42' \, 30''=10{\cdot}495800$, $\text{L}\tan 8° \, 28' \, 55{\cdot}5''=9{\cdot}173582$.

$$\tan^2\frac{\phi}{2}=\frac{b-c}{b+c}=\frac{1-\frac{c}{b}}{1+\frac{c}{b}} \qquad \text{[\textit{See} xix. 1 (ii.)]}$$

$$\frac{1-\cos\phi}{1+\cos\phi}=\frac{1-\frac{c}{b}}{1+\frac{c}{b}}. \qquad \text{(xxxvi. 10.)}$$

$$\cos\phi=\frac{c}{b}=\frac{10}{11}$$

$$\text{L}\cos\phi-10=1-\log 11$$

$$\text{L}\cos\phi=11-1{\cdot}041393=9{\cdot}958607$$

$$\phi=24^\circ\ 37'\ 12''$$

$$\therefore \frac{\phi}{2}=12^\circ\ 18'\ 36''.$$

$$\tan\frac{B-C}{2}=\tan^2 12^\circ\ 18'\ 36''\times\cot 17^\circ\ 42'\ 30''$$

$$\text{L}\tan\frac{B-C}{2}-10=2\ \text{L}\tan 12^\circ\ 18'\ 36''-20+\text{L}\cot 17^\circ\ 42'\ 30''-10$$

$$\text{L}\tan\frac{B-C}{2}=18{\cdot}677782-20+10{\cdot}495800$$

$$=9{\cdot}173582$$

$$\frac{B-C}{2}=8^\circ\ 28'\ 56{\cdot}5''$$

$$\frac{B+C}{2}=90^\circ-17^\circ\ 42'\ 30''$$

$$=72^\circ\ 17'\ 30''$$

$$\therefore B=80^\circ\ 46'\ 26{\cdot}5''.$$

$$C=63^\circ\ 48'\ 33{\cdot}5''.$$

4. **Prop.** Given two sides and the angle opposite one of them, viz., a, b, A.

$$\sin B=\frac{b\sin A}{a},\ \text{gives } B.$$

$$C=180^\circ-(A+B).$$

$$c=\frac{b\sin C}{\sin B}\quad\text{or}\quad\frac{a\sin C}{\sin A}.$$

Here the angle B is found from its sine; and since $\sin B=\sin(180^\circ-B)$ there *may be* two angles, one acute, the other obtuse, which satisfy the equation $\sin B=\frac{b\sin A}{a}$.

Hence **the ambiguous case of the solution of triangles.**

(i.) $b \sin A > a$, $\sin B > 1$, which is impossible.
No triangle exists with the given parts. No solution.

(ii.) $b \sin A = a$, $\sin B = 1$, $B = 90°$, a right-angled triangle.
One solution.

(iii.) $b \sin A < a$, $\sin B < 1$, and there *may be* two solutions, one of the angles being the supplement of the other, *i.e.* one acute, the other obtuse.

(*a*) If $b < a$,
then $B < A$, and since A is given, B must be acute.
Only one solution.

(β) If $b > a$,
then $B > A$, and in this case A must be acute, and B may therefore be either acute or obtuse; and there are two solutions.

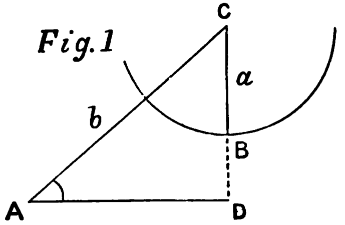
I. Prop.—Ambiguous case in the solution of triangles, illustrated Geometrically.

ABC is the triangle with the given parts a, b and A.

With centre C and radius a, describe a circle, and draw the perpendicular CD, which $= b \sin A$.

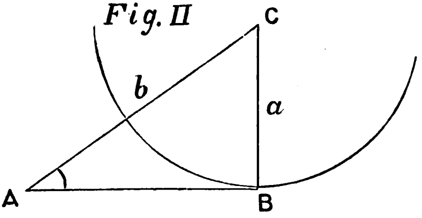
(i.) $b \sin A > a$, the circle does not cut AD, and the construction fails. No solution. (*Fig. I.*)

Fig. 1



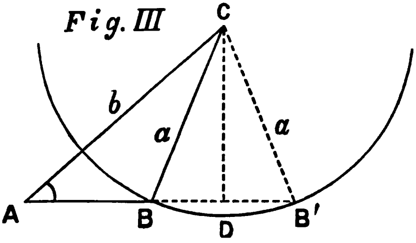
(ii.) $b \sin A = a$, the circle touches AB in B, and the triangle is right-angled. One solution. (*Fig. II.*)

Fig. II



(iii.) $b \sin A < a$, we have *Figs. III.* and *IV.*

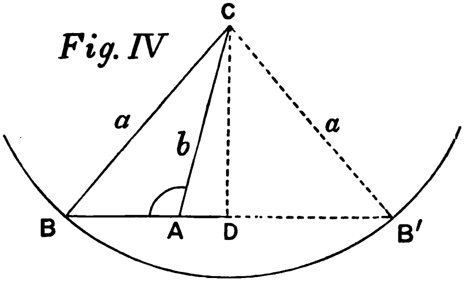
Fig. III



(α) In *Fig. IV*, $a > b$, and we have one triangle ABC with the given parts ; the triangle $AB'C$ having the sides a, b and the angle $180° - A$.

One solution.

Fig. IV



(β) In *Fig. III*, $a < b$, we have two triangles, ABC, $AB'C$, both having the given parts a, b, A.

This is the case in which there are two solutions, when

$$b \sin A < a, \text{ and } b > a.$$

II.—Algebraical method.

Prop. Given a, b, A, discuss the ambiguous case by finding the side c.

$$a^2=b^2+c^2-2bc\cos A$$

$$\therefore\ c^2-2bc\cos A+(b\cos A)^2=a^2-b^2(1-\cos^2 A)=a^2-b^2\sin^2 A$$

$$\therefore\ c=b\cos A\pm\sqrt{a^2-b^2\sin^2 A},$$

(i.) If $b\sin A>a$, the expression under the root is negative, and the values of c are imaginary. No solution.

(ii.) If $b\sin A=a$, the roots are equal, $c=b\cos A$. One solution, a right-angled triangle, $B=90°$.

(iii.) If $b\sin A<a$, the roots are $b\cos A+\sqrt{a^2-b^2\sin^2 A}$, and $b\cos A-\sqrt{a^2-b^2\sin^2 A}$; the latter is only admissible if $b\cos A>\sqrt{a^2-b^2\sin^2 A}$

If $b^2(\cos^2 A+\sin^2 A)>a^2$

If $b^2>a^2$

If $b>a$.

Hence there are two solutions when $b\sin A<a$, and $b>a$.

N.B.—If c_1, c_2 are the roots of $c^2-2bc\cos A+b^2-a^2=0$, by the theory of quadratics,

$$\left.\begin{aligned}c_1+c_2&=2b\cos A\\ c_1c_2&=b^2-a^2\end{aligned}\right\}$$

These results are useful and should be remembered.

III. In the solution of triangles there can be no ambiguity, except when an angle is determined by the sine or cosecant, and in no case whatever, when the triangle has one right angle. (cxxix. 11.)

Since the angle A of a triangle is $<180°$, if $\cos A$ be $+$, A is $<90°$; if $\cos A$ is $-$, A is $>90°$; the same is true of the tan, cot, and sec.

If $\sin A$ be given, it cannot be $-$, since A is $< 180°$, and since $\sin A = \sin(180° - A)$, there are two values of A, one acute, the other obtuse, corresponding to a given value of $\sin A$; and unless there be some other condition to show whether A is obtuse or acute, it is clear that A will admit of two different values.

If the angle be a right angle or obtuse, the other two angles are both acute, and there can be no ambiguity.

EXAMPLES.

Ex. 1. Given $A = 40°$, $a = 140{\cdot}5$, $b = 170{\cdot}6$, find B and C; given $L\sin 40° = 9{\cdot}8080675$, $L\sin 51°\ 18' = 9{\cdot}8923342$, diff. for $1' = {\cdot}0001012$, $\log 1405 = 3{\cdot}1476763$, $\log 1706 = 3{\cdot}2319790$.

$$\sin B = \frac{b \sin A}{a} = \frac{170{\cdot}6 \sin 40°}{140{\cdot}5} \qquad \text{(x. 5.)}$$

$$\begin{aligned} L\sin B &= \log 170{\cdot}6 - \log 140{\cdot}5 + L\sin 40° \\ &= 2{\cdot}2319790 - 2{\cdot}1476763 + 9{\cdot}8080675 \end{aligned}$$

$$\left.\begin{aligned} &= 9{\cdot}8923702 \\ L\sin 51°\ 18' &= 9{\cdot}8923342 \end{aligned}\right\} 360 \qquad \frac{360}{1012} \times 60'' = 21''$$

$$\begin{aligned} \therefore\ B &= 51°\ 18'\ 21'', \text{ or } 128°\ 41'\ 39''. \\ A &= 40°. \\ C &= 88°\ 41'\ 39'', \text{ or } 11°\ 18'\ 21''. \end{aligned}$$

Ex. 2. If a, b are the two given sides ($a<b$), A the given angle, and C, C' the two values of the ambiguous angle, then

$$\cos A \cos B = \sin\frac{C+C'}{2} \sin\frac{C-C'}{2}.$$

$$\begin{aligned} \sin\frac{C+C'}{2} \sin\frac{C-C'}{2} &= \frac{1}{2}\left\{\cos C' - \cos C\right\} \qquad \text{(xli. 9.)} \\ &= \frac{1}{2}\left\{\cos(B-A) + \cos(A+B)\right\} \\ &= \cos A \cos B. \end{aligned}$$

Since $C = \pi - (A+B)$, $C' = \pi - \{A + (\pi - B)\} = B - A$.

Ex. 3. Show that the difference between the two values of C in the ambiguous case when a, b, A are given, is $2\cos^{-1}\left\{\frac{b\sin A}{a}\right\}$.

[*See* Fig. III. in 4 (I.)] ACB and ACB' are the two values of C

$$ACB' - ACB = BCB' = 2BCD$$

$$\cos BCD = \frac{CD}{CB} = \frac{b\sin A}{a} \qquad \text{(lxviii. 5.)}$$

$$BCD = \cos^{-1}\left\{\frac{b\sin A}{a}\right\}$$

$$\therefore\ 2BCD = 2\cos^{-1}\left\{\frac{b\sin A}{a}\right\}.$$

Ex. 4. Find the sum of the areas of the two triangles which satisfy the conditions of the problem in the ambiguous case.

$$\text{Sum of areas} = \frac{1}{2}b\sin A\ (c_1 + c_2)$$

$$= \frac{1}{2}b\sin A\ (2b\cos A) = b^2\sin A\cos A.$$

Ex. 5. In the ambiguous case, if A, B, C, and A, B', C' be the angles of the triangles in the two solutions, prove that $\frac{\sin C}{\sin B} + \frac{\sin C'}{\sin B'} = 2\cos A$.

From Fig. I. Prop. I.

$$\frac{\sin C}{\sin B} + \frac{\sin C'}{\sin B'} = \frac{c_1}{b} + \frac{c_2}{b} \qquad \text{(cxxvii. 7.)}$$

$$= \frac{c_1 + c_2}{b} = \frac{2b\cos A}{b} = 2\cos A.$$

Ex. 6. In the ambiguous case, prove that

$$c_1^2 - 2c_1c_2\cos 2A + c_2^2 = 4a^2\cos^2 A.$$

$$\text{Left-hand member} = c_1^2 + c_2^2 - 2c_1c_2(2\cos^2 A - 1)$$

$$= \overline{c_1 + c_2}|^2 - 4c_1c_2\cos^2 A$$

$$= 4b^2\cos^2 A - 4(b^2 - a^2)\cos^2 A$$

$$= 4a^2\cos^2 A.$$

XVI.

Measurement of Heights and Distances.

1. Def. **The "Angle of Elevation" of a point above the eye of an observer, is the angle made with the horizontal plane by a straight line drawn from the observer's eye to the point.**

If the point lie below the eye of the observer, the angle is called the "**Angle of Depression.**"

These angles are measured by the **Theodolite.**

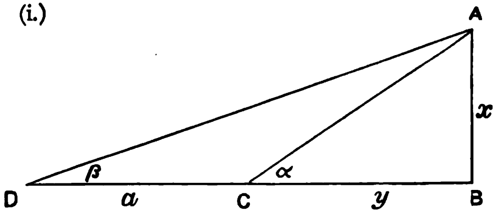
2. The angle subtended at the eye of an observer by a line joining two distant objects is measured by a **Sextant.**

3. The points of the "**Mariner's Compass**" should be carefully learnt. The Quadrant is divided into eight equal parts, each called a *point*, and the points are subdivided into half and quarter points.

One point $=11^\circ\ 15'$.
Half point $=\ 5^\circ\ 37'\ 30''$.
Quarter point$=\ 2^\circ\ 48'\ 45''$.

4. Prop. Find the height and distance of an inaccessible object on a horizontal plane.

(i.)



Let x be the height of the object AB, and y its distance from C, the nearest accessible point.

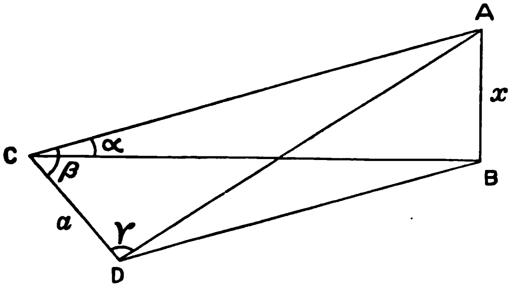
At C observe the $\angle ACB=\alpha$, measure $CD=a$, and at D observe the angle $ADB=\beta$.

$$\text{Then } \frac{AD}{a}=\frac{\sin ACD}{\sin DAC}=\frac{\sin(180^\circ-\alpha)}{\sin(\alpha-\beta)}=\frac{\sin\alpha}{\sin(\alpha-\beta)}, \qquad AD=\frac{a\sin\alpha}{\sin(\alpha-\beta)},$$

$$\therefore\ x=AD\sin\beta=\frac{a\sin\alpha\sin\beta}{\sin(\alpha-\beta)}.$$

$$y=x\cot\alpha=\frac{a\sin\beta\sin\alpha}{\sin(\alpha-\beta)}\times\frac{\cos\alpha}{\sin\alpha}=\frac{a\cos\alpha\sin\beta}{\sin(\alpha-\beta)}.$$

(ii.) If the line CD cannot be measured towards the object.



At C observe

$ACB=\alpha$,

$ACD=\beta$,

measure $CD=a$,

and at D observe

$ADC=\gamma$.

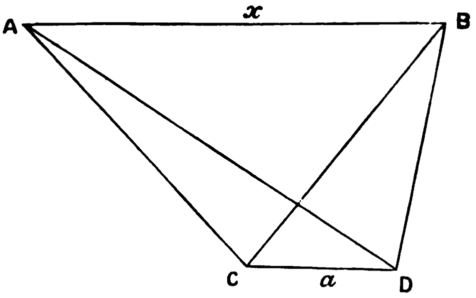
Then $\dfrac{AC}{a}=\dfrac{\sin\gamma}{\sin(180^\circ-(\beta+\gamma))}=\dfrac{\sin\gamma}{\sin(\beta+\gamma)}$, $AC=\dfrac{a\sin\gamma}{\sin(\beta+\gamma)}$

$$\therefore\ x=AC\sin\alpha=\frac{a\sin\alpha\sin\gamma}{\sin(\beta+\gamma)}.$$

$$y=AC\cos\alpha=\frac{a\cos\alpha\sin\gamma}{\sin(\beta+\gamma)}.$$

5. **Prop.** Find the distance between two visible inaccessible objects.

Let x be the distance between the two objects A, B. Let C, D be the two accessible points from which A and B are visible.



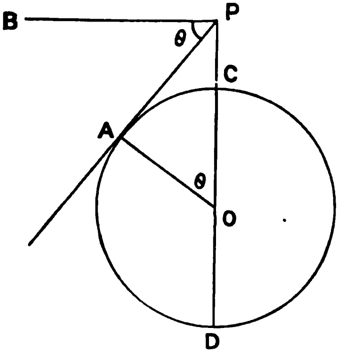
At C observe the angles ACB, BCD, and if necessary ACD.

Measure $CD=a$, and at D observe the angles ADC, ADB.

Then in the triangle ACD, we know two angles and a side, $\therefore AC$ may be found. In the triangle BCD, we know two angles and a side, $\therefore CB$ may be found.

Lastly in the triangle ABC we have AC, BC, and the included angle ACB, from which x may be found.

6. PROP. Dip of the Horizon.



If O be the centre of the earth, P a point at a height PC $(=h)$ above the surface, PA a tangent to the surface at A; then A is on the terrestrial horizon; and APB, which is the angle of depression of the most distant point on the horizon seen from P, is called the "Dip of the horizon."

$$POA = \text{complement of } OPA = \theta$$

$$h = PC = OP - OC = OA\ (\sec\theta - 1) = r\,\frac{1-\cos\theta}{\cos\theta}, \text{ if } r = \text{radius of earth}$$

$$\therefore\ r = \frac{h\cos\theta}{1-\cos\theta}.$$

$$PA = OA\tan\theta = \frac{h\cos\theta}{1-\cos\theta}\tan\theta = \frac{h\sin\theta}{1-\cos\theta} = h\cot\frac{\theta}{2}.$$

Also $PA^2 = DP\,.\,PC$ (Eucl. iii. 36.) (cxviii. 3.)

$$= h\ (h + CD)$$

$$= h\ (h + 2r).$$

Since h is very small compared with $2r$

$$PA^2 = 2hr, \text{ approximately.}$$

If n = the number of miles in PA, h = the feet in PC, and $r = 4000$ miles nearly.

$$= \frac{PA^2}{2r} = \frac{(5280n)^2}{8000\times 5280}$$

$$= \frac{5280n^2}{8000}$$

$$= \frac{5{\cdot}28n^2}{8}$$

$$= {\cdot}66n^2$$

$$= \frac{2n^2}{3} \text{ nearly,}$$

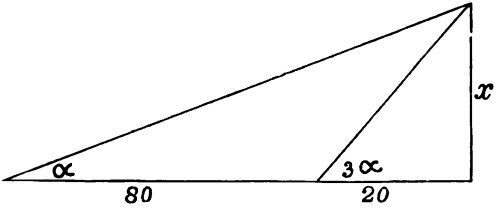
$$\text{or,}\quad n^2 = \frac{3h}{2}.$$

or, Three times the height in feet of the place of observation above the sea is equal to twice the square of the distance of the horizon in miles if $n=1$ mile, $h=\frac{2}{3}$ ft. $=8$ ins.

(*i.e.*) An object less than 8 inches high will be visible from a point on the earth 1 mile distant.

EXAMPLES.

Ex. 1. The angle of elevation of a tower at a distance of 20 yards from its foot is three times as great as the angle of elevation 100 yards from the same point; show that the height of the tower is $\frac{300}{\sqrt{7}}$ feet.



$$\tan a=\frac{x}{100}, \quad \tan 3a=\frac{x}{20}$$

$$\tan 3a=\frac{3\tan a-\tan^3 a}{1-3\tan^2 a}$$

$$\frac{x}{20}=\frac{\frac{3x}{100}-\frac{x^3}{100^3}}{1-\frac{3x^2}{100^2}}$$

$$\frac{1}{20}=\frac{\frac{3}{100}-\frac{x^2}{100^3}}{1-\frac{3x^2}{100^2}}=\frac{3(100^2)-x^2}{100\,(100^2-3x^2)} \qquad \text{(xxix. 9.)}$$

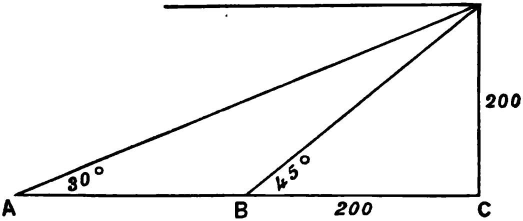
$$5\,(100^2-3x^2)=3(100^2)-x^2$$

$$2(100^2)=14x^2$$

$$x^2=\frac{100^2}{7}$$

$$\therefore\ x=\frac{100}{\sqrt{7}}\text{ yards}=\frac{300}{\sqrt{7}}\text{ feet.}$$

Ex. 2. An observer on the top of a cliff, 200 feet above the sea-level, observes the angles of depression of two ships at anchor, in the same vertical plane with the observer, to be 45° and 30° respectively; find the distance of the ships from each other.



$BC = 200$

$AC = 200 \cot 30°$

$x = 200 (\cot 30° - 1)$

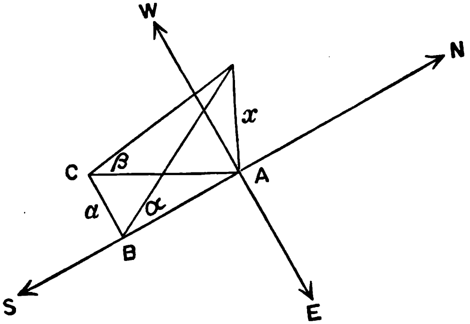
$= 200 (\sqrt{3} - 1)$

$= 200 (\cdot 73205)$

$= 146\cdot 4.$

(xxxix. 7.)

Ex. 3. A man standing due south of a spire, finds the angular elevation of its summit to be α. He then walks to a point a yards due west of his former position, and finds the elevation to be β. Show that the height of the spire in yards is $\dfrac{a \sin \alpha \sin \beta}{\sqrt{\sin(\alpha - \beta) \sin(\alpha + \beta)}}$.



$$AB = x \cot \alpha, \quad AC = x \cot \beta$$

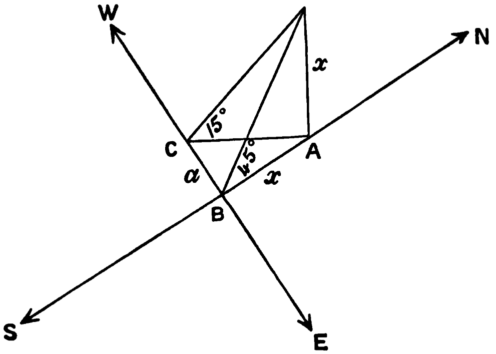
$$x^2 \cot^2 \beta = a^2 + x^2 \cot^2 \alpha$$

$$x^2 = \frac{a^2}{\cot^2 \beta - \cot^2 \alpha} = \frac{a^2 \sin^2 \alpha \sin^2 \beta}{\sin^2 \alpha \cos^2 \beta - \cos^2 \alpha \sin^2 \beta}$$

$$= \frac{a^2 \sin^2 \alpha \sin^2 \beta}{\sin(\alpha + \beta) \sin(\alpha - \beta)} \qquad \text{(lix. 11.)}$$

$$\therefore x = \frac{a \sin \alpha \sin \beta}{\sqrt{\sin(\alpha + \beta) \sin(\alpha - \beta)}}.$$

Ex. 4. The angular elevation of a steeple at a place due south of it is 45°, and at another place, due west of the former station, the elevation is 15° ; show that the height of the steeple is $\frac{a}{2}(3^{\frac{1}{4}}-3^{-\frac{1}{4}})$, a being the distance between the places.



$$AB=x$$

$$AC=x\cot 15^\circ$$

$$x^2\cot^2 15^\circ=a^2+x^2 \qquad \text{(viii. 7.)}$$

$$x^2(7+4\sqrt{3})=a^2+x^2$$

$$x^2(6+4\sqrt{3})=a^2$$

$$x^2=\frac{a^2}{6+4\sqrt{3}}=\frac{a^2(4\sqrt{3}-6)}{48-36}$$

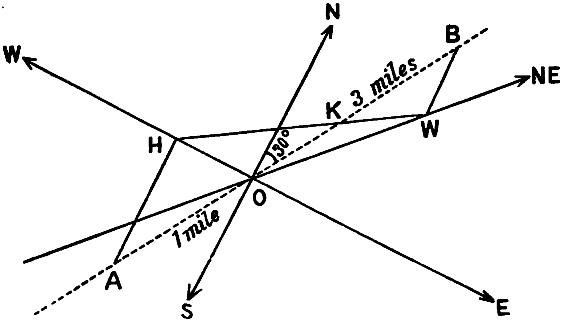
$$=\frac{a^2(2-\sqrt{3})}{2\sqrt{3}}$$

$$\therefore x=\frac{a\frac{\sqrt{3}-1}{\sqrt{2}}}{\sqrt{2}\sqrt[4]{3}}=\frac{a\sqrt{3}-1}{2\sqrt[4]{3}}=\frac{a}{2}(3^{\frac{1}{4}}-3^{-\frac{1}{4}}$$

Ex. 5. A man walking along a straight road, which runs in a direction 30° east of north, notes when he is due south of a certain house. When he has walked a mile further, he observes that the house lies due west, and that a windmill on the opposite side of the road is N.E. of him. Three miles further on, he finds that he is due north of the windmill.

Find the distance between the house and the windmill, and show that the line joining them makes with the road an angle $\tan^{-1}\dfrac{48-25\sqrt{3}}{11}$

H the house, *W* the windmill. Let $HKO = a$, $OH = \dfrac{1}{2}$



$$\frac{OW}{OB} = \frac{\sin 30^\circ}{\sin 135^\circ}, \quad \frac{OW}{3} = \frac{1}{\sqrt{2}} \quad \therefore OW = \frac{3}{\sqrt{2}}.$$

$$HW^2 = OH^2 + OW^2 - 2OH.OW \cos 135^\circ$$

$$= \frac{1}{4} + \frac{9}{2} + 2 \cdot \frac{1}{2} \frac{3}{\sqrt{2}} \frac{1}{\sqrt{2}}$$

$$= \frac{1}{4} + \frac{9}{2} + \frac{3}{2} = \frac{25}{4} \qquad \text{(xxxv. 12.)}$$

$$\therefore HW = \frac{5}{2} = 2\frac{1}{2} \text{ miles.}$$

$$\frac{HK}{HO} = \frac{\sin 120^\circ}{\sin a}, \quad HK = \frac{\sqrt{3}}{4 \sin a}$$

$$\frac{KW}{OW} = \frac{\sin 15^\circ}{\sin a}, \quad KW = \frac{3(\sqrt{3}-1)}{4 \sin a}$$

$$\frac{\sqrt{3} + 3(\sqrt{3}-1)}{4 \sin a} = \frac{5}{2}$$

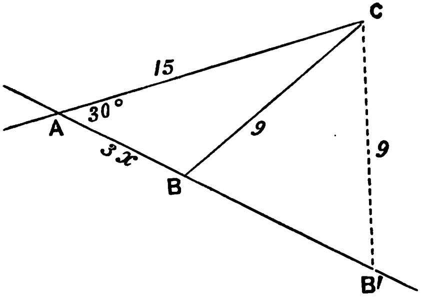
$$\therefore \sin a = \frac{4\sqrt{3}-3}{10}.$$

$$\therefore a = \sin^{-1}\left\{\frac{4\sqrt{3}-3}{10}\right\}$$

$$= \tan^{-1}\left\{\frac{4\sqrt{3}-3}{3\sqrt{3}+4}\right\} = \tan^{-1}\left\{\frac{48-25\sqrt{3}}{11}\right\}$$

Ex. 6. Two straight roads intersect at an angle of 30° : from the point of junction two pedestrians *A* and *B* start at the same time, *A* walking along one of the roads at the rate of 5 miles an hour, *B* walking uniformly along the other road. At the end of 3 hours *A* and *B* are 9 miles apart. Show that there are two rates at which *B* may walk to fulfil the condition, and determine the slower rate of the two.

Given $\log 2 = {\cdot}3010300$, $\log 3 = {\cdot}4771213$, $\log 8{\cdot}0154 = {\cdot}9039248$,
$L\sin 56^\circ\,27' = 9{\cdot}9208555$, $L\sin 56^\circ\,26' = 9{\cdot}9207717$,
$L\sin 26^\circ\,26'\,33'' = 9{\cdot}6486522$.



$$\sin B = \frac{15}{9}\sin 30^\circ = \frac{5}{6}$$

$$\begin{aligned} L\sin B &= 10 + \log 5 - \log 6 \\ &= 10 + 1 - \log 2 - \log 3 - \log 2 && \text{(xci. 4.)} \\ &= 11 - 2\log 2 - \log 3 \end{aligned}$$

$$\begin{array}{rll} & = 9{\cdot}9208187 & \\ L\sin 56^\circ\,26' & = 9{\cdot}9207717 & \big\}\ 470 \\ L\sin 56^\circ\,27' & = 9{\cdot}9208555 & \big\}\ 838 \qquad \frac{470}{838}\times 60'' = 33'' \end{array}$$

$$\therefore B = 56^\circ\,26'\,33'' \text{ or } 123^\circ\,33'\,27''.$$

$$C = 180^\circ - (123^\circ\,33'\,27'' + 30^\circ) = 26^\circ\,26'\,33''.$$

$$3x = \frac{15\sin 26^\circ\,26'\,33''}{\sin 56^\circ\,26'\,33''}$$

$$\begin{aligned} \log 3x &= \log 3 + \log 5 + L\sin 26^\circ\,26'\,33'' - L\sin 56^\circ\,26'\,33'' \\ &= {\cdot}9039248 \end{aligned}$$

$$3x = 8{\cdot}0154$$

$$\therefore x = 2{\cdot}6718 \text{ miles an hour.}$$

XVII.

Properties of Triangles.

A, B, C are the angles of any triangle.

a, b, c the opposite sides.

$s=\frac{a+b+c}{2}$, the semiperimeter, S=the area (Eucl. i. 41).

r=radius of the inscribed circle (Eucl. iv. 4).

R=radius of circumscribed circle (Eucl. iv. 5).

r_a, r_b, r_c the radii of the escribed circles, touching the sides a, b, c respectively, and the other sides produced.

1. Prop. Area of triangle

$$=\frac{1}{2}\text{ perpendicular}\times\text{base}$$

$$=\frac{1}{2}\,ab\sin C=\frac{1}{2}\,ac\sin B=\frac{1}{2}\,bc\sin A$$

$$=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\frac{1}{2}\,\frac{a^2\sin B\sin C}{\sin A}=\frac{1}{2}\,\frac{b^2\sin A\sin C}{\sin B}=\frac{1}{2}\,\frac{c^2\sin A\sin B}{\sin C}.$$

2. Prop. $r=\frac{S}{s}=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$

$$=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}=\frac{b\sin\frac{C}{2}\sin\frac{A}{2}}{\cos\frac{B}{2}}=\frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}.$$

Length of the tangent to the inscribed circle from the angle $A=s-a$, from the angle $B=s-b$, from the angle $C=s-c$.

3. **Prop.** $R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4S}$.

4. **Prop.** $r_a=\frac{S}{s-a}=s\tan\frac{A}{2}=\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$.

$$r_b=\frac{S}{s-b}=s\tan\frac{B}{2}=\frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}.$$

$$r_c=\frac{S}{s-c}=s\tan\frac{C}{2}=\frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}.$$

Length of the tangent to the escribed circle touching the side a, from the angle $A=s$, from the angle $B=s-c$, from the angle $C=s-b$.

N.B.—The line joining the centres of the inscribed circle and the escribed circle touching the side a passes through the angle A.

5. **Prop.**

$$r_a\cot\frac{A}{2}=r_b\cot\frac{B}{2}=r_c\cot\frac{C}{2}=r\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

$$=4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}. \qquad \text{(xli. 8.)}$$

6. **Prop.** Distance between the centres of inscribed and circumscribed circles$=\sqrt{R^2-2Rr}$.

7. **Prop.** Distance between the centre of the inscribed circle and the escribed circle touching the side $a = \sqrt{R^2 + 2Rr_a}$.

$$\text{Also, Distance} = \frac{r_a - r}{\sin\frac{A}{2}} = \frac{1}{\sin\frac{A}{2}}\left\{\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} - \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}\right\}$$

$$= \frac{a\cos\frac{B+C}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{a}{\cos\frac{A}{2}} = a\sec\frac{A}{2}.$$

8. **Prop.** Distance between the centres of the escribed circles are $\dfrac{a}{\sin\frac{A}{2}}, \dfrac{b}{\sin\frac{B}{2}}, \dfrac{c}{\sin\frac{C}{2}}$. (civ. 3.)

9. **Prop.** Area of a Quadrilateral inscribed in a circle $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$,

where a, b, c, d are the sides, and s = half the perimeter.

$$\text{Radius of circumscribing circle} = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}.$$

10. The following properties should be proved—

- (i.) The lines joining the angular points A, B, C of a triangle with A', B', C', the middle points of the opposite sides, meet in a point, which is the centre of gravity of the triangle.
- (ii.) The lines AD, BE, CF drawn from the angular points of a triangle perpendicular to the opposite sides, meet in a point (P), which is called the **Orthocentre** of the triangle ABC.

Def. The triangle *DEF*, joining the feet of the perpendiculars is called the Pedal triangle of *ABC*.

Def. The Nine Points Circle is the circle which passes through the feet of the perpendiculars.

(iii.) The Nine Points Circle

(α) Passes through the points A', B', C'.

(β) Passes through the middle points of PA, PB, PC.

(γ) Touches the inscribed and escribed circles.

(δ) Has a radius $=\frac{R}{2}$.

(iv.) If O, O_1, O_2, O_3 be the centres of the inscribed and escribed circles

(α) ABC is the pedal triangle of $O_1 O_2 O_3$.

(β) O is the orthocentre of $O_1 O_2 O_3$.

(γ) Radius of circle circumscribing $O_1 O_2 O_3 = 2R$.

(v.) If a', b', c', and A', B', C', be the sides and angles of the Pedal triangle of ABC, then

$$a'=a\cos A,\quad b'=b\cos B,\quad c'=c\cos C,$$

$$A'=\pi-2A,\quad B'=\pi-2B,\quad C'=\pi-2C.$$

Hence also $A=\frac{\pi}{2}-\frac{A'}{2}$, and $a=a'\sec A=a'\operatorname{cosec}\frac{A'}{2}$.

$$B=\frac{\pi}{2}-\frac{B'}{2},\text{ and } b=b'\sec B=b'\operatorname{cosec}\frac{B'}{2}.$$

$$C=\frac{\pi}{2}-\frac{C'}{2},\text{ and } c=c'\sec C=c'\operatorname{cosec}\frac{C'}{2}.$$

And since corresponding to every triangle there is a pedal triangle, it follows that **any formula connecting the sides and angles of a triangle is also true if for the sides we write $a\cos A$, $b\cos B$, $c\cos C$, and for the angles $\pi-2A$, $\pi-2B$, $\pi-2C$:** or again, if for the sides we write $a\operatorname{cosec}\frac{A}{2}$, $b\operatorname{cosec}\frac{B}{2}$, $c\operatorname{cosec}\frac{C}{2}$, and for the angles $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$, $\frac{\pi}{2}-\frac{C}{2}$. (*See* ci. 2.)

EXAMPLES.

Ex. 1. Prove that $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$. (xvi. 7.)

Left-hand member$=\frac{s-a+s-b+s-c}{S}=\frac{3s-(a+b+c)}{S}=\frac{s}{S}=\frac{1}{r}$.

Ex. 2. Prove that $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

(xl. 9.)

$$r=\frac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}},\quad\text{and}\quad R=\frac{c}{2\sin C}=\frac{c}{4\sin\frac{C}{2}\cos\frac{C}{2}}.$$

$$\therefore\ \frac{r}{R}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Ex. 3. Find the area of a triangle whose sides are ·9, 1·2, 1·5 respectively.

$a=\ \cdot 9$

$b=1\cdot 2$

$c=1\cdot 5$ (xxxix. 9.)

$3\cdot 6$

$s=1\cdot 8$

Area$=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{1\cdot 8\times\cdot 9\times\cdot 6\times\cdot 3}$

$$=\sqrt{\frac{18\times 9\times 6\times 3}{10000}}=\frac{1}{100}\sqrt{18^2\times 3^2}=\frac{18\times 3}{100}=\cdot 54.$$

Ex. 4. If the lengths of the sides of a triangle be 242, 1212, and 1450 yards, show that the area is 6 acres.

$$\begin{aligned} a &= 242 \\ b &= 1212 \\ c &= 1450 \\ \hline &\,2904 \\ s &= 1452 \end{aligned} \qquad \text{(xxxvi. 8.)}$$

$$\begin{aligned} \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{1452 \times 1210 \times 240 \times 2} \\ &= \sqrt{10^2 \times 121^2 \times 24^2} \\ &= 121 \times 24 \times 10 \\ &= 29040 \text{ square yards} \\ &= \frac{29040}{4840} \text{ acres} = 6 \text{ acres.} \end{aligned}$$

Ex. 5. The sides of a triangle are as $2:3:4$, show that the radii of the escribed circles are as $\frac{1}{5}:\frac{1}{3}:1$.

$$\begin{aligned} a &= 2 \\ b &= 3 \\ c &= 4 \\ \hline &\,9 \\ s &= \frac{9}{2} \end{aligned} \qquad \text{(lx. 10.)}$$

$$\begin{aligned} r_a : r_b : r_c &= \frac{S}{s-a} : \frac{S}{s-b} : \frac{S}{s-c} \\ &= \frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c} \\ &= \frac{1}{\frac{5}{2}} : \frac{1}{\frac{3}{2}} : \frac{1}{\frac{1}{2}} \\ &= \frac{1}{5} : \frac{1}{3} : 1. \end{aligned}$$

Ex. 6. Prove that the area formed by joining the points of contact of the inscribed circle equals $2r^2 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$.

Join the angular points of the new triangle with the centre of the inscribed circle.

The angles at the centre are respectively $\pi - B$, $\pi - C$, $\pi - A$.

$$\begin{aligned} \text{Area} &= \frac{1}{2} r^2 \sin(\pi - A) + \frac{1}{2} r^2 \sin(\pi - B) + \frac{1}{2} r^2 \sin(\pi - C) \\ &= \frac{1}{2} r^2 (\sin A + \sin B + \sin C) \\ &= \frac{1}{2} r^2 \left(4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}\right) \qquad \text{(ii. 6.)} \\ &= 2r^2 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}. \end{aligned}$$

Ex. 7. Show that the length of a side of an equilateral triangle inscribed in a circle is to that of a side of a square inscribed in the same circle as $\surd 3 : \surd 2$.

Let a and a' be the sides of the triangle and square respectively,

$$\text{Then } R = \frac{a}{2\sin\frac{\pi}{3}} = \frac{a'}{2\sin\frac{\pi}{4}}$$

$$\begin{aligned} \therefore\ a : a' &= \sin\frac{\pi}{3} : \sin\frac{\pi}{4} \qquad \text{(xiv. 8.)} \\ &= \frac{\surd 3}{2} : \frac{1}{\surd 2} \\ &= \surd 3 : \surd 2. \end{aligned}$$

Ex. 8. The sum of the diameters of the inscribed and circumscribed circles is $a \cot A + b \cot B + c \cot C$.

$$\begin{aligned} R + r &= R\left(1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}\right) \qquad \text{(by Ex. 2.)} \\ &= R(\cos A + \cos B + \cos C) \qquad \text{(\textit{See} Chap. VII. Prop. 6.)} \\ &= \frac{a\cos A}{2\sin A} + \frac{b\cos B}{2\sin B} + \frac{c\cos C}{2\sin C} \\ &= \frac{1}{2}(a \cot A + b \cot B + c \cot C) \end{aligned}$$

$$\therefore\ 2R + 2r = a \cot A + b \cot B + c \cot C.$$

Ex. 9. The sides of a triangle are in A.P, and its area is to that of an equilateral triangle of the same perimeter as 3 : 5. Find the ratio of the sides and the value of the largest angle.

Let $a-b$, a, $a+b$ denote the sides,

$$\text{then } s=\frac{3a}{2}. \qquad \text{(cxxvii. 11.)}$$

$$\text{Area}=\sqrt{\frac{3a}{2}\left(\frac{a}{2}+b\right)\frac{a}{2}\left(\frac{a}{2}-b\right)}=\frac{a}{4}\sqrt{3(a^2-4b^2)}.$$

Area of equilateral triangle of the same perimeter,

$$\textit{i.e.}, \text{ with side } a=\frac{\sqrt{3}a^2}{4}.$$

$$\frac{a}{4}\sqrt{3\ (a^2-4b^2)}:\frac{\sqrt{3}a^2}{4}::3:5$$

$$25\ (a^2-4b^2)=9a^2$$

$$16a^2=100b^2$$

$$\therefore a=\frac{5b}{2}.$$

The sides are in the ratio $\frac{3b}{2}, \frac{5b}{2}, \frac{7b}{2}$, or 3 : 5 :: 7.

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{9+25-49}{30}=-\frac{15}{30}=-\frac{1}{2}$$

$$\therefore C=120^\circ.$$

Ex. 10. If a, b, c be the sides of a triangle, and α, β, γ the perpendiculars drawn upon them from the centre of the circumscribing circle

$$\text{prove that} \qquad 4\left\{\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}\right\}=\frac{abc}{\alpha\beta\gamma}.$$

Draw OD perpendicular to BC, then $BOD=A$,

$$\tan A=\frac{a}{2\alpha},\ \tan B=\frac{b}{2\beta},\ \tan C=\frac{c}{2\gamma}.$$

When $A+B+C=\pi$, $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

$$\therefore\ \frac{a}{2\alpha}+\frac{b}{2\beta}+\frac{c}{2\gamma}=\frac{abc}{8\alpha\beta\gamma} \qquad \text{(xx. 5.)}$$

$$\therefore\ 4\left\{\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}\right\}=\frac{abc}{\alpha\beta\gamma}.$$

XVIII.

Regular Polygons, Inscribed and Circumscribed Circles, Area of Circle.

r = radius of circle inscribed in a regular polygon of n sides, each side being $=a$.

R = radius of circumscribed circle.

1. Prop. $a=2R\sin\dfrac{\pi}{n}=2r\tan\dfrac{\pi}{n}\cdot$

Perimeter of polygon $=2nR\sin\dfrac{\pi}{n}=2nr\tan\dfrac{\pi}{n}\cdot$

2. Prop. Circumference of circle lies between $2nr\sin\dfrac{\pi}{n}$ and $2nr\tan\dfrac{\pi}{n}$,

i.e., between $2\pi r\dfrac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}$ and $2\pi r\dfrac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}$,

and when $n=\infty$, $\dfrac{\pi}{n}=0$, and $\dfrac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}=\dfrac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}=1$, (*See* Chap. xx. Prop. 2.)

$$\therefore \text{circumference}=2\pi r.$$

3. Prop. Area of Polygon

$$=\frac{n}{2}R^2\sin\frac{2\pi}{n}=nr^2\tan\frac{\pi}{n}=\frac{na^2}{4}\cot\frac{\pi}{n}\cdot$$

4. PROP. Area of circle lies between—

$$\frac{1}{2}nr^2 \sin\frac{2\pi}{n} \text{ and } nr^2 \tan\frac{\pi}{n},$$

$$i.e., \text{ between } \pi r^2 \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} \text{ and } \pi r^2 \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}},$$

$$\text{and when } n=\infty,\ \frac{\pi}{n}=0,\ \text{ and } \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}=\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}=1,$$

$$\therefore \text{ area of circle} = \pi r^2.$$

5. PROP. Area of sector of circle $=\frac{1}{2}$ arc × radius $=\frac{r^2\theta}{2}$,

where θ is the circular measure of the angle of the sector.

6. PROP.

$$\frac{\text{side of regular polygon of } n \text{ sides inscribed in circle}}{\text{side of regular polygon of } n \text{ sides circumscribing circle}}=\cos\frac{\pi}{n}.$$

$$\frac{\text{area of inscribed polygon}}{\text{area of circumscribed polygon}}=\cos^2\frac{\pi}{n}.$$

EXAMPLES.

Ex. 1. An equilateral triangle and a regular hexagon have the same perimeter. Show that the areas of their inscribed circles are as 4 : 9.

Let $6a$ = perimeter of each,

$\therefore$ side of triangle $=2a$, side of hexagon $=a$.

Area of inscribed circles are as $\pi r^2 : \pi r_1^2$

$$=\frac{(2a)^2}{4\tan^2\frac{\pi}{3}} : \frac{a^2}{4\tan^2\frac{\pi}{6}} \qquad \text{(xvi. 8.)}$$

$$=\frac{4}{3} : \frac{1}{\frac{1}{3}}$$

$$=4 : 9.$$

Ex. 2. Prove that the square of the side of an inscribed pentagon equals the sum of the squares of the sides of a hexagon and decagon inscribed in the same circle.

Let p, h, d represent a side of the figures respectively.

$$\text{Since } R = \frac{a}{2\sin\frac{\pi}{n}} \left\{ \begin{array}{l} p=2R\sin\frac{\pi}{5} \\ h=2R\sin\frac{\pi}{6} \\ d=2R\sin\frac{\pi}{10} \end{array} \right. \qquad \text{(xiii. 7.)}$$

$$\therefore h^2+d^2=4R^2\left\{\sin^2\frac{\pi}{6}+\sin^2\frac{\pi}{10}\right\}$$

$$=4R^2\left\{\frac{1}{4}+\frac{6-2\sqrt{5}}{16}\right\}$$

$$=4R^2\frac{10-2\sqrt{5}}{16}$$

$$=4R^2\frac{5-\sqrt{5}}{8}$$

$$=4R^2\sin^2\frac{\pi}{5}$$

$$=p^2.$$

Ex. 3. Prove that $R+r=\frac{a}{2}\cot\frac{\pi}{2n}$. (xxxiv. 9.)

$$R+r=\frac{a}{2\sin\frac{\pi}{n}}+\frac{a}{2\tan\frac{\pi}{n}}=\frac{a}{2}\,\frac{1+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}$$

$$=\frac{a}{2}\,\frac{2\cos^2\frac{\pi}{2n}}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}=\frac{a}{2}\cot\frac{\pi}{2n}.$$

XIX.

Subsidiary Angles.

An expression in the form of a product or quotient is said to be "adapted to logarithmic computation."

If an expression consist of terms connected by + and −, it may be adapted to logarithms by means of a subsidiary angle.

DEF. **An angle introduced into an expression in order to adapt it to logarithmic computation is called a Subsidiary Angle.**

The value of the sine and cosine of an angle ranges from 0 to 1; we may, therefore, assume $\sin\phi$ or $\cos\phi=\frac{a}{b}$, if a is not $>b$.

The value of the tangent ranges from 0 to ∞; we may, therefore, assume for any real values of a and b that $\tan\phi=\frac{a}{b}$.

1. (i.) $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$

$$=\frac{1-\frac{b}{a}}{1+\frac{b}{a}}\cot\frac{C}{2},\quad \text{let } \frac{b}{a}=\tan\phi$$

$$=\frac{1-\tan\phi}{1+\tan\phi}\cot\frac{C}{2}$$

$$=\tan\left(\frac{\pi}{4}-\phi\right)\cot\frac{C}{2}$$

$$\therefore \text{L}\tan\frac{A-B}{2}=\text{L}\tan\left(\frac{\pi}{4}-\phi\right)+\text{L}\cot\frac{C}{2}-10,$$

where $\text{L}\tan\phi=10+\log b-\log a$.

Ex. 2. Pro
equals the su
inscribed in t

Let p, h, d

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$$

$$=\frac{1-\frac{b}{a}}{1+\frac{b}{a}}\cot\frac{C}{2},\quad \text{let } \frac{b}{a}=\cos\phi$$

$$=\frac{1-\cos\phi}{1+\cos\phi}\cot\frac{C}{2} \qquad \text{(xxxvi. 10.)}$$

$$=\tan^2\frac{\phi}{2}\cot\frac{C}{2}$$

$$\therefore\ \mathrm{L}\tan\frac{A-B}{2}=2\left(\mathrm{L}\tan\frac{\phi}{2}-10\right)+\mathrm{L}\cot\frac{C}{2},$$

where $\mathrm{L}\cos\phi=10+\log b-\log a.$

2. (i.) $c^2=a^2+b^2-2ab\cos C$

$$=a^2+b^2-2ab\left\{2\cos^2\frac{C}{2}-1\right\}$$

$$=(a+b)^2-4ab\cos^2\frac{C}{2} \qquad \text{(lxi. 7.)}$$

$$=(a+b)^2\left\{1-\frac{4ab}{(a+b)^2}\cos^2\frac{C}{2}\right\}$$

Let $\sin^2\phi=\dfrac{4ab}{(a+b)^2}\cos^2\dfrac{C}{2}$, which is possible, since $(a+b)^2>4ab.$

$$c^2=(a+b)^2(1-\sin^2\phi)$$

$$=(a+b)^2\cos^2\phi$$

$$c=(a+b)\cos\phi$$

$$\therefore\ \log c=\log(a+b)+\mathrm{L}\cos\phi-10,\quad \text{and}$$

$$\therefore\ 2(\mathrm{L}\sin\phi-10)=2\log 2+\log a+\log b-2\log(a+b)+2\left(\mathrm{L}\cos\frac{C}{2}-10\right)$$

$$\text{or}\quad 2\,\mathrm{L}\sin\phi=2\log 2+\log a+\log b-2\log(a+b)+2\,\mathrm{L}\cos\frac{C}{2}.$$

(ii.) $$c^2=a^2+b^2-2ab\cos C$$
$$=a^2+b^2-2ab\left(1-2\sin^2\frac{C}{2}\right)$$
$$=(a-b)^2+4ab\sin^2\frac{C}{2}$$
$$=(a-b)^2\left(1+\frac{4ab}{(a-b)^2}\sin^2\frac{C}{2}\right)$$

Let $\tan^2\phi=\frac{4ab}{(a-b)^2}\sin^2\frac{C}{2}$,

$$c^2=(a-b)^2(1+\tan^2\phi)$$
$$=(a-b)^2\sec^2\phi$$
$$c=(a-b)\sec\phi$$

$\therefore\ \log c=\log(a-b)+\text{L}\sec\phi-10$, and

$$2(\text{L}\tan\phi-10)=2\log 2+\log a+\log b-2\log(a-b)+2\left(\text{L}\sin\frac{C}{2}-10\right)$$

or $2\,\text{L}\tan\phi=2\log 2+\log a+\log b-2\log(a-b)+2\,\text{L}\sin\frac{C}{2}$.

(iii.) $$c^2=a^2+b^2-2ab\cos C$$
$$=(a^2+b^2)\left(\cos^2\frac{C}{2}+\sin^2\frac{C}{2}\right)-2ab\left(\cos^2\frac{C}{2}-\sin^2\frac{C}{2}\right)$$
$$=(a+b)^2\sin^2\frac{C}{2}+(a-b)^2\cos^2\frac{C}{2}=(a+b)^2\sin^2\frac{C}{2}\left\{1+\left(\frac{a-b}{a+b}\cot\frac{C}{2}\right)^2\right\}$$

Let $\tan\phi=\frac{a-b}{a+b}\cot\frac{C}{2}$, (lxii. 6.

$$c^2=(a+b)^2\sin^2\frac{C}{2}(1+\tan^2\phi)=(a+b)^2\sin^2\frac{C}{2}\sec^2\phi$$
$$c=(a+b)\sin\frac{C}{2}\sec\phi$$

$\therefore\ \log c=\log(a+b)+\text{L}\sin\frac{C}{2}+\text{L}\sec\phi-20$,

and $\text{L}\tan\phi=\log(a-b)-\log(a+b)+\text{L}\cot\frac{C}{2}$.

3. (i.) $x=a+b$, where a and b are positive.

$$=a\left(1+\frac{b}{a}\right),\ \text{let }\tan^2\phi=\frac{b}{a}$$
$$=a(1+\tan^2\phi)$$
$$=a\sec^2\phi$$

$\therefore\ \log x=\log a+2\,\text{L}\sec\phi-20$,

and $2\,\text{L}\tan\phi=20+\log b-\log a$.

(ii.) $x=a+b$, where a and b are not both positive.

$$=a\left(1+\frac{b}{a}\right), \text{ let } \tan\phi=\frac{b}{a}$$

$$=a\,(1+\tan\phi)$$

$$=\frac{a\sqrt{2}}{\cos\phi}\left(\frac{1}{\sqrt{2}}\cos\phi+\frac{1}{\sqrt{2}}\sin\phi\right)$$

$$=\frac{a\sqrt{2}}{\cos\phi}\sin\left(\phi+\frac{\pi}{4}\right)$$

$\therefore\ \log x=\log a+\frac{1}{2}\log 2+\text{L}\sin\left(\phi+\frac{\pi}{4}\right)-\text{L}\cos\phi,$

and $\text{L}\tan\phi=10+\log b-\log a.$

4. $x=a-b$

$$=a\left(1-\frac{b}{a}\right), \quad \text{let } \cos^2\phi=\frac{b}{a}$$

$$=a\,(1-\cos^2\phi)=a\sin^2\phi$$

$\therefore\ \log x=\log a+2\,\text{L}\sin\phi-20$, and $2\,\text{L}\cos\phi=20+\log b-\log a.$

5. $x=\sqrt{a^2+b^2}$

$$=a\sqrt{1+\frac{b^2}{a^2}}, \quad \text{let } \tan\phi=\frac{b}{a}$$

$$=a\sqrt{1+\tan^2\phi}\quad=a\sec\phi$$

$\therefore\ \log x=\log a+\text{L}\sec\phi-10$, and $\text{L}\tan\phi=\log b-\log a.$

6. $x=\sqrt{a+b}+\sqrt{a-b}.$

$$=\sqrt{a}\left\{\sqrt{1+\frac{b}{a}}+\sqrt{1-\frac{b}{a}}\right\}$$

Let $\cos\phi=\frac{b}{a}$, since $a>b$,

$$x=\sqrt{a}\left\{\sqrt{1+\cos\phi}+\sqrt{1-\cos\phi}\right\}$$

$$=\sqrt{a}\left\{\sqrt{2}\cos\frac{\phi}{2}+\sqrt{2}\sin\frac{\phi}{2}\right\}$$

$$=2\sqrt{a}\left\{\frac{1}{\sqrt{2}}\cos\frac{\phi}{2}+\frac{1}{\sqrt{2}}\sin\frac{\phi}{2}\right\}$$

$$=2\sqrt{a}\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right), \text{ or } 2\sqrt{a}\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$$

$\therefore\ \log x=\log 2+\frac{1}{2}\log a+\text{L}\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-10,$

and $\text{L}\cos\phi=10+\log b-\log a.$

XX.

Construction of Trigonometrical Tables.

1. Prop. If θ be the circular measure of a positive angle less than a right angle, $\sin\theta$, θ, $\tan\theta$ are in ascending order of magnitude.

Or, $\theta > \sin\theta$ and $< \tan\theta$.

2. Prop. The limit of $\frac{\sin\theta}{\theta}$ when θ is indefinitely diminished is 1.

Or, $\sin\theta = \theta$, very nearly.

3. Prop. The limit of $\frac{\tan\theta}{\theta}$ when θ is indefinitely diminished is 1.

Or, $\tan\theta = \theta$, very nearly.

4. Prop. The limit of $\frac{\sin n^\circ}{n}$ when n is indefinitely diminished is the circular measure of 1°.

The limit of $\frac{\sin n'}{n}$ when n is indefinitely diminished is the circular measure of 1′; and *generally*—

The limit of the ratio of the sine of an angle to the angle, when the angle is indefinitely diminished is the circular measure of the unit, whatever be the unit of angular measurement.

5. Prop. The limit of $\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\dots\dots\cos\frac{\theta}{2^n}$ is $\frac{\sin\theta}{\theta}$, when $n = \infty$.

6. **Prop.** If θ be a positive angle, $< \frac{\pi}{2}$, $\sin\theta > \theta - \frac{\theta^3}{4}$.

$$
\begin{aligned}
\sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\
&> 2\left\{\frac{\theta}{2}\cos\frac{\theta}{2}\right\}\cos\frac{\theta}{2}\\
&> \theta\cos^2\frac{\theta}{2}\\
&> \theta\left(1-\sin^2\frac{\theta}{2}\right)\\
\textit{à fortiori}\quad &> \theta\left(1-\frac{\theta^2}{4}\right)\\
&> \theta-\frac{\theta^3}{4}.
\end{aligned}
$$

$$
\begin{aligned}
\because \tan\theta &> \theta\\
\tan\frac{\theta}{2} &> \frac{\theta}{2}\\
\therefore \sin\frac{\theta}{2} &> \frac{\theta}{2}\cos\frac{\theta}{2}.
\end{aligned}
$$

$$
\begin{aligned}
\because \sin\theta &< \theta\\
\sin\frac{\theta}{2} &< \frac{\theta}{2}\\
\therefore \sin^2\frac{\theta}{2} &< \frac{\theta^2}{4}.
\end{aligned}
$$

$\therefore$ $\sin\theta$ is $< \theta$ and $> \theta - \frac{\theta^3}{4}$.

7. **Prop.** If θ be a positive angle, less than $\frac{\pi}{2}$,

$$\cos\theta > 1-\frac{\theta^2}{2} < 1-\frac{\theta^2}{2}+\frac{\theta^4}{16}.$$

$$
\begin{aligned}
\cos\theta &= 1-2\sin^2\frac{\theta}{2}\\
&> 1-2\left(\frac{\theta^2}{4}\right)\\
&> 1-\frac{\theta^2}{2}.
\end{aligned}
$$

$$
\begin{aligned}
\because \sin\theta &< \theta\\
\sin\frac{\theta}{2} &< \frac{\theta}{2}\\
\therefore \sin^2\frac{\theta}{2} &< \frac{\theta^2}{4}.
\end{aligned}
$$

$$
\begin{aligned}
\cos\theta &= 1-2\sin^2\frac{\theta}{2}\\
&< 1-2\left\{\frac{\theta}{2}-\frac{\theta^3}{32}\right\}^2\\
&< 1-2\left\{\frac{\theta^2}{4}-\frac{\theta^4}{32}+\frac{\theta^6}{32^2}\right\}\\
&< 1-\frac{\theta^2}{2}+\frac{\theta^4}{16}-\frac{\theta^6}{512}\\
\textit{à fortiori}\quad &< 1-\frac{\theta^2}{2}+\frac{\theta^4}{16}.
\end{aligned}
$$

$$
\begin{aligned}
\sin\theta &> \theta-\frac{\theta^3}{4}\\
\therefore \sin\frac{\theta}{2} &> \frac{\theta}{2}-\frac{\theta^3}{32}.
\end{aligned}
$$

8. **Prop.** If θ be the circular measure of a positive angle $<\frac{\pi}{2}$, prove that $\sin\theta > \theta - \frac{\theta^3}{6}$; $\cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$.

$$\text{(i.)}\quad \sin\theta = 3\sin\frac{\theta}{3} - 4\sin^3\frac{\theta}{3}$$

$$\therefore 3\sin\frac{\theta}{3} = 3^2\sin\frac{\theta}{3^2} - 4.3\sin^3\frac{\theta}{3^2}\quad \left\{\begin{array}{l}\text{write } \frac{\theta}{3} \text{ for } \theta\text{, and}\\ \text{multiply by 3}\end{array}\right.$$

similarly $3^2\sin\frac{\theta}{3^2} = 3^3\sin\frac{\theta}{3^3} - 4.3^2\sin^3\frac{\theta}{3^3}$

etc. = etc.

$$3^{n-1}\sin\frac{\theta}{3^{n-1}} = 3^n\sin\frac{\theta}{3^n} - 4.3^{n-1}\sin^3\frac{\theta}{3^n}.$$

By addition,

$$\sin\theta = 3^n\sin\frac{\theta}{3^n} - 4\left\{\sin^3\frac{\theta}{3} + 3\sin^3\frac{\theta}{3^2} + \ldots + 3^{n-1}\sin^3\frac{\theta}{3^n}\right\}.$$

Now $\sin\theta < \theta$, $\therefore \sin\frac{\theta}{3} < \frac{\theta}{3}$,

$$\therefore \sin\theta > 3^n\sin\frac{\theta}{3^n} - \frac{4\theta^3}{3^3}\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \ldots\right)$$

if $n = \infty$, $3^n\sin\frac{\theta}{3^n} = \dfrac{\sin\frac{\theta}{3^n}}{\frac{\theta}{3^n}} = \theta$, in the limit,

and $\frac{4\theta^3}{3^3}\left(1 + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\right) = \frac{4\theta^3}{3^3}\dfrac{1}{1 - \frac{1}{3^2}} = \frac{\theta^3}{6}$

$$\therefore \sin\theta > \theta - \frac{\theta^3}{6}.$$

(ii.) We have $\sin\theta > \theta - \frac{\theta^3}{6}$

$$\therefore \sin\frac{\theta}{2} > \frac{\theta}{2} - \frac{\theta^3}{48}$$

$$1 - 2\sin^2\frac{\theta}{2} < 1 - 2\left(\frac{\theta}{2} - \frac{\theta^3}{48}\right)^2$$

$$\therefore \cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

9. **Prop.** If θ be the circular measure of a positive angle $< \frac{\pi}{2}$, prove that $\tan\theta > \theta + \frac{\theta^3}{3}$.

$$\text{We have } \sin\theta > \theta - \frac{\theta^3}{6}$$

$$\text{And } \cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$$

$$\therefore \tan\theta > \frac{\theta - \frac{\theta^3}{6}}{1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}}$$

$$> \theta + \frac{\theta^3}{3} + \frac{\frac{\theta^5}{8}\left(1 - \frac{\theta^2}{9}\right)}{1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}}, \text{ by division;}$$

now $1 - \frac{\theta^2}{9}$ is positive, if $\theta < 3$, *à fortiori* if $\theta < \frac{\pi}{2}$,

$1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$ is positive, if $\theta^2 < 6 + \sqrt{12}$, *à fortiori*, if $\theta < \frac{\pi}{2}$,

$\therefore$ if $\theta < \frac{\pi}{2}$, the fraction is positive,

$$\therefore \tan\theta > \theta + \frac{\theta^3}{3}.$$

10. **Prop.** Show how to find the value of sine 10″.

Let θ be the circular measure of 10″,

$$\text{then } \theta = \frac{10\pi}{180 \times 60 \times 60} = \frac{\pi}{64800} = \cdot 000\,048\,481\,368 \,\ldots\ldots$$

Since $\sin\theta < \theta$, and $> \theta - \frac{\theta^3}{4}$,

$$\sin 10'' < \theta, \text{ and } > \theta - \frac{1}{4}(\cdot 000\,0484\,\ldots)^3$$

$$< \theta, \text{ and } > \theta - \cdot 000\,000\,000\,000\,028\,\ldots\ldots$$

Hence $\sin 10'' =$ the circular measure of $10''$, correct to 12 places of decimals.

Then $\cos 10''$ may be found to be $\cdot 999\,999\,998\,8248 \dots$

And since $\sin(n+1)\theta + \sin(n-1)\theta = 2\sin n\theta\cos\theta$,

if $\theta = 10''$, putting $n = 1, 2, \dots$ we get

$$\sin 20'' = 2\sin 10''\cos 10''$$
$$\sin 30'' + \sin 10'' = 2\sin 20''\cos 10''$$
$$\text{etc.} = \text{etc.}$$

in this way we can form a table of the sines and cosines of angles which differ by $10''$.

11. When the sines of angles up to 45° have been found, we have

$$\sin(45^\circ + A) - \sin(45^\circ - A) = \sqrt{2}\sin A,$$

by which those for the remainder of the quadrant may be deduced.

When the sines of angles up to 60° have been found, we have

$$\sin(60^\circ + A) - \sin(60^\circ - A) = \sin A$$

for the remainder of the quadrant.

Having completed a table of sines, the cosines are known, since $\cos A = \sin(90^\circ - A)$. Then the tangents may be found; and when the tangents up to 45° have been found, we may use

$$\tan(45^\circ + A) = \tan(45^\circ - A) + 2\tan 2A$$

to find the tangents of angles from 45° to 90°.

Having found the sines, cosines, and tangents, the other trig-ratios may also be found.

12. Formulæ used to test the accuracy of the calculated sines or cosines of angles are called Formulæ of Verification.

Euler's Formulæ:—

$$\sin(36^\circ + A) - \sin(36^\circ - A) - \sin(72^\circ + A) + \sin(72^\circ - A) = \sin A.$$
$$\cos(36^\circ + A) + \cos(36^\circ - A) - \cos(72^\circ + A) - \cos(72^\circ - A) = \cos A.$$

In which we give to A any value, and take from the tables the sines and cosines of the angles involved, and these values must satisfy the above equations.

Legendre's Formula :—

$$\sin(90^\circ - A) = \cos A$$

$$= \sin(54^\circ + A) + \sin(54^\circ - A) - \sin(18^\circ + A) - \sin(18^\circ - A).$$

For example, if $A = 11^\circ$, taking from the tables the sines involved, it should be true that

$$\sin 47^\circ - \sin 25^\circ - \sin 83^\circ + \sin 61^\circ = \sin 11^\circ,$$

$$\text{or,}\quad \sin 79^\circ = \sin 65^\circ + \sin 43^\circ - \sin 29^\circ - \sin 7^\circ.$$

EXAMPLES.

Ex. 1. Given $\frac{\sin\theta}{\theta} = \frac{1013}{1014}$, prove that the value of θ is 4° 24′ nearly.

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6}, \text{ nearly.}$$

$$1 - \frac{1}{1014} = 1 - \frac{\theta^2}{6} \qquad \text{(lxxi. 1.)}$$

$$\frac{\theta^2}{6} = \frac{1}{1014}$$

$$\theta^2 = \frac{1}{169}, \quad \theta = \frac{1}{13}$$

$$\therefore \theta = \frac{1}{13} \text{ of } 57^\circ{\cdot}295 \;.\;.\;.\;.\; = 4^\circ\, 24' \text{ nearly.}$$

Ex. 2. Given $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$, show that θ contains 3° nearly.

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6}, \text{ nearly.}$$

$$1 - \frac{1}{2166} = 1 - \frac{\theta^2}{6}$$

$$\frac{\theta^2}{6} = \frac{1}{2166}$$

$$\theta^2 = \frac{1}{361}, \quad \theta = \frac{1}{19}$$

$$\therefore \theta = \frac{1}{19} \text{ of } 57^\circ \text{ nearly} = 3^\circ \text{ nearly.}$$

XXI.

Napierian Logarithms. Exponential Theorem.

1. **Prop.** Expand e^x in a series of ascending powers of x.

By the Binomial Theorem,

$$\left(1+\frac{1}{n}\right)^{nx}=1+nx\frac{1}{n}+\frac{nx\,(nx-1)}{\underline{|2}}\,\frac{1}{n^2}+\dots$$

$$+\frac{nx\,(nx-1)\,\dots\,(nx-r+1)}{\underline{|r}}\,\frac{1}{n^r}+\dots$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\underline{|2}}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{|3}}+\dots\frac{x\left(x-\frac{1}{n}\right)\dots\left(x-\frac{r-1}{n}\right)}{\underline{|r}}+\dots$$

Now as n increases $\frac{1}{n}$, $\frac{2}{n}$, . . . diminish,

and $x-\frac{1}{n}$, $x-\frac{2}{n}$, . . . increase,

So that as n increases, the value of $\left(1+\frac{1}{n}\right)^{nx}$ continually increases, and when n becomes infinite, $\frac{1}{n}$, $\frac{2}{n}$, . . . vanish.

Hence the limit, when n becomes ∞, of

$$\left(1+\frac{1}{n}\right)^{nx}=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\dots\frac{x^r}{\underline{|r}}+\dots$$

If $x=1$, we have, when $n=\infty$

$$\left(1+\frac{1}{n}\right)^{n}=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\dots+\frac{1}{\underline{|r}}+\dots$$

and this series is denoted by e.

Now $\left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x}=\left(1+\frac{1}{n}\right)^{nx}$,

$$\therefore\ e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\dots\frac{x^r}{\underline{|r}}+\dots$$

2. Find the Arithmetical Value of *e*.

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \ldots$$

$$\begin{aligned}
1 + 1 + \frac{1}{\underline{|2}} &= 2{\cdot}5 \\
\frac{1}{\underline{|3}} &= {\cdot}1666666666 \\
\frac{1}{\underline{|4}} &= {\cdot}0416666666 \\
\frac{1}{\underline{|5}} &= {\cdot}0083333333 \\
\frac{1}{\underline{|6}} &= {\cdot}0013888888 \\
\frac{1}{\underline{|7}} &= {\cdot}0001984126 \\
\frac{1}{\underline{|8}} &= {\cdot}0000248015 \\
\frac{1}{\underline{|9}} &= {\cdot}0000027557 \\
\frac{1}{\underline{|10}} &= {\cdot}0000002755 \\
\therefore\ e &= 2{\cdot}7182818
\end{aligned}$$

the correct value of the base of the Napierian system of logarithms to seven places of decimals, which may be determined to any degree of accuracy by taking more terms of the series.

3. The Quantity *e* is > 2 and < 3, and is therefore convergent.

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \ldots,$$

e is obviously > 2,

$$\begin{aligned}
\text{and} &< 1 + \left[1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\right] \\
&< 1 + \frac{1}{1 - \frac{1}{2}} \\
&< 1 + 2 \\
&< 3.
\end{aligned}$$

4. Prop. The Quantity e is incommensurable.

For, if possible, let it $=\frac{m}{n}$, where m, n are + integers,

$$\therefore \frac{m}{n}=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots+\frac{1}{\underline{|n}}+\frac{1}{\underline{|n+1}}+\ldots$$

Multiply both sides by $\underline{|n}$,

$$m\,\underline{|n-1}=\text{integer}+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\ldots$$

$$\text{Now } \frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots$$

$$\text{is} > \frac{1}{n+1} \text{ and } < \frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\ldots$$

$$< \frac{\frac{1}{n+1}}{1-\frac{1}{n+1}}$$

$$< \frac{1}{n}$$

this series is therefore a fraction,

Therefore $m\,\underline{|n-1}=$ integer + fraction

(*i.e.*) integer = integer + fraction

which is impossible,

Therefore e is incommensurable.

5. **Prop.** The Quantity e cannot be the root of a quadratic equation with rational coefficients.

If possible, suppose that

$$ae^2+be+c=0, \text{ where } a, b, c \text{ are integers.}$$

$$\therefore\ ae+ce^{-1}+b=0$$

$$a\left\{1+1+\frac{1}{|\underline{2}}+\frac{1}{|\underline{3}}+\ldots\frac{1}{|\underline{m}}+\frac{1}{|\underline{m+1}}+\ldots\right\}+c\left\{1-1+\frac{1}{|\underline{2}}-\ldots\right\}+b=0$$

Multiply by $|\underline{m}$,

$$\text{Integer}+a\left\{\frac{1}{m+1}+\frac{1}{(m+1)(m+2)}+\ldots\right\}\pm c\left\{\frac{1}{m+1}-\frac{1}{(m+1)(m+2)}+\ldots\right\}=0$$

$$\text{or } \frac{a}{m+1}\left\{1+\frac{1}{m+2}+\ldots\right\}\pm\frac{c}{m+1}\left\{1-\frac{1}{m+2}+\ldots\right\}=\text{integer}$$

the expression on the left hand is $<\frac{a}{m}\pm\frac{c}{m}$

which is <1, if m be taken large enough.

Hence we have,

Fraction = Integer,

which is impossible.

6. **Prop.** Expand a^x in ascending powers of x.

The Exponential Theorem.

We have proved, $e^z=1+z+\frac{z^2}{|\underline{2}}+\frac{z^3}{|\underline{3}}+\ \ldots\ldots$

Let $a^x=e^z$,

Then $z=\log_e a^x=x\log_e a$,

Substitute for z in e^z $\therefore\ a^x=1+x\log_e a+\frac{x^2(\log_e a)^2}{|\underline{2}}+\frac{x^3(\log_e a)^3}{|\underline{3}}+\ \ldots$

7. Prop. Expand $\log_e(1+x)$ in ascending powers of x.

We have—

$$a^z = 1 + z\log_e a + \frac{z^2(\log_e a)^2}{\underline{|2}} + \ldots.$$

Write $1+a$ for a,

$$(1+a)^z = 1 + z\log_e(1+a) + \frac{z^2\{\log_e(1+a)\}^2}{\underline{|2}} + \ldots.$$

By the Binomial Theorem,

$$(1+a)^z = 1 + za + \frac{z(z-1)}{\underline{|2}}a^2 + \frac{z(z-1)(z-2)}{\underline{|3}}a^3 + \ldots$$

Equate the co-efficients of z in these two expansions of $(1+a)^z$,

$$\log_e(1+a) = a - \frac{a^2}{2} + \frac{1.2a^3}{\underline{|3}} - \frac{1.2.3a^4}{\underline{|4}} + \ldots.$$

$$= a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \ldots.$$

Write x for a,

$$\therefore \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.$$

a series which is only convergent, when $x > -1$ and < 1; and unless x is very small, it converges very slowly, and is of little practical use for numerical calculation.

Write $-x$ for x.

$$\text{Log}_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots.$$

N.B.—If we required to expand $\{\log_e(1+x)\}^2$ we should take the co-efficient of z^2 from the two expansions of $(1+a)^z$.

8. Prop. Series for Logarithms of numbers to base e.

We have seen that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots.$$

$$\log(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots.\right)$$

$$i.e., \log\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots.\right)$$

$$\text{Let } m=\frac{1+x}{1-x}, \quad \text{then } x=\frac{m-1}{m+1},$$

$$\text{(i.)} \therefore \log m=2\left\{\frac{m-1}{m+1}+\frac{1}{3}\left(\frac{m-1}{m+1}\right)^3+\frac{1}{5}\left(\frac{m-1}{m+1}\right)^5+\ldots\right\}$$

$$\text{Let } \frac{m}{n}=\frac{1+x}{1-x}, \quad \text{then } x=\frac{m-n}{m+n},$$

$$\text{(ii.)} \therefore \log\frac{m}{n}=2\left\{\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\ldots\right\}$$

let $m=n+1$,

$$\text{(iii.)} \therefore \log(n+1)-\log n=2\left\{\frac{1}{2n+1}+\frac{1}{3}\left(\frac{1}{2n+1}\right)^3+\frac{1}{5}\left(\frac{1}{2n+1}\right)^5+\ldots\right\}$$

The series (iii.) is rapidly convergent, and gives the Logarithm of a number from that of the next inferior number.

9. PROP. Show how to form a Table of Logarithms to base *e*.

It may be shown that—

$$\text{Log}_e(n+1)-\log_e n=2\left\{\frac{1}{2n+1}+\frac{1}{3}\left(\frac{1}{2n+1}\right)^3+\frac{1}{5}\left(\frac{1}{2n+1}\right)^5+\ldots\right\}$$

From which equation, by actual calculation, we can find the value of $\log_e(n+1)$ when $\log_e n$ is known.

If $n=1$, $\log_e n=0$,

$$\therefore \log_e 2=2\left\{\frac{1}{3}+\frac{1}{3}\left(\frac{1}{3}\right)^3+\frac{1}{5}\left(\frac{1}{3}\right)^5+\ldots\right\}=\cdot 6931471\ldots$$

Hence $\log_e 2$ may be calculated to any degree of accuracy.

$$\text{Then } \log_e 3=\log_e 2+\frac{1}{2}\left(\frac{1}{5}+\frac{1}{3}\left(\frac{1}{5}\right)^3+\frac{1}{5}\left(\frac{1}{5}\right)^5+\ldots\right)$$
$$=1\cdot 0986123\ldots$$

Thus the logs of successive numbers to base *e* may be found by a series of laborious calculations.

10. PROP. Show how to form a Table of Logarithms to base 10.

The modulus (μ) for transferring logarithms from base *e* to base 10

$$\frac{1}{\log_e 10}=\frac{1}{2\cdot 30258509\ldots}=\cdot 43429448\ldots$$

By multiplying each of the logarithms to base *e* by this quantity we obtain the logarithms of the same numbers to base 10;

$$\text{Log}_{10} 2=\mu \log_e 2=\cdot 4342944 \times \cdot 6931472=\cdot 3010300$$
$$\text{Log}_{10} 3=\mu \log_e 3=\cdot 4342944 \times 1\cdot 0986123=\cdot 4771213 \text{ etc.} \ldots$$

11. **Prop.** Compare the advantages of base 10 and base e.

(i.) As 10 is the radix of our system of notation, it is proved that the characteristic can be found by inspection; and by always making the mantissa positive, we need only register the mantissæ of the logarithms of whole numbers, as the logarithms of decimals with the same significant digits will differ only in characteristic:

Thus base 10 shortens the Tables.

(ii.) The Napierian system is more adapted to theoretical investigations in which we may have to expand the logarithms of different expressions:

Thus base e simplifies Formulæ.

12. **Prop.** Find the limit of $\left(\cos\dfrac{\theta}{n}\right)^n$, when $n=\infty$.

$$
\begin{aligned}
\text{Let } x &= \left(\cos\frac{\theta}{n}\right)^n \\
&= \left(1-\sin^2\frac{\theta}{n}\right)^{\frac{n}{2}} \\
\log x &= \frac{n}{2}\log\left(1-\sin^2\frac{\theta}{n}\right) \\
&= -\frac{n}{2}\left\{\sin^2\frac{\theta}{n}+\frac{1}{2}\sin^4\frac{\theta}{n}+\frac{1}{3}\sin^6\frac{\theta}{n}+\dots\right\} \\
&< -\frac{n}{2}\left\{\sin^2\frac{\theta}{n}+\sin^4\frac{\theta}{n}+\sin^6\frac{\theta}{n}+\dots\right\} \\
&< -\frac{n}{2}\,\frac{\sin^2\frac{\theta}{n}}{1-\sin^2\frac{\theta}{n}} \\
&< -\frac{n}{2}\tan^2\frac{\theta}{n} \\
&< -\frac{\theta^2}{2n}\left(\frac{\tan\frac{\theta}{n}}{\frac{\theta}{n}}\right)^2, \text{ when } n=x,\ \frac{\tan\frac{\theta}{n}}{\frac{\theta}{n}}=1 \\
&< -\frac{\theta^2}{2n}.
\end{aligned}
$$

But when $n=\infty$, $\dfrac{\theta^2}{2n}=0$

$$\log x=0; \qquad \therefore\ x=1.$$

13. **Prop.** Find the limit of $\left\{\dfrac{\sin\dfrac{\theta}{n}}{\dfrac{\theta}{n}}\right\}^n$, when $n=\infty$,

$\sin\dfrac{\theta}{n}$, $\dfrac{\theta}{n}$, $\tan\dfrac{\theta}{n}$ are in ascending order of magnitude.

$$\frac{\sin\dfrac{\theta}{n}}{\dfrac{\theta}{n}} < 1, \text{ and } > \frac{\sin\dfrac{\theta}{n}}{\tan\dfrac{\theta}{n}}$$

$$< \quad 1, \quad > \cos\frac{\theta}{n}$$

$$\left\{\frac{\sin\dfrac{\theta}{n}}{\dfrac{\theta}{n}}\right\}^n < \quad 1^n, \quad > \left\{\cos\frac{\theta}{n}\right\}^n$$

but the limit of $\left(\cos\dfrac{\theta}{n}\right)^n$ is 1.

$\therefore$ the limit of $\left\{\dfrac{\sin\dfrac{\theta}{n}}{\dfrac{\theta}{n}}\right\}^n$ is 1.

Examples.

Ex. 1. Prove the following theorem :—

$$\text{Log}\left\{(1+N)^{\frac{1+N}{2}}(1-N)^{\frac{1-N}{2}}\right\}=\frac{N^2}{1.2}+\frac{N^4}{3.4}+\frac{N^6}{5.6}+\ldots\ldots$$

$$\text{Left hand member}=\frac{1+N}{2}\log(1+N)+\frac{1-N}{2}\log(1-N)$$

$$=\frac{1}{2}\log(1+N)(1-N)+\frac{N}{2}\log\frac{1+N}{1-N} \qquad \text{(cxix. 3.)}$$

$$=\frac{1}{2}\log(1-N^2)+\frac{N}{2}\log\frac{1+N}{1-N}$$

$$=\frac{1}{2}\left\{-N^2-\frac{N^4}{2}-\frac{N^6}{3}-\frac{N^8}{4}\ldots\ldots\right\}$$

$$+\frac{N}{2}\cdot 2\left\{N+\frac{N^3}{3}+\frac{N^5}{5}+\ldots\ldots\right\}$$

$$=\left(N^2-\frac{N^2}{2}\right)+\left(\frac{N^4}{3}-\frac{N^4}{4}\right)+\left(\frac{N^6}{5}-\frac{N^6}{6}\right)+\ldots\ldots$$

$$=\frac{N^2}{1.2}+\frac{N^4}{3.4}+\frac{N^6}{5.6}+\ldots\ldots$$

Ex. 2. Show that the series $1+2+\frac{3}{1.2}+\frac{4}{1.2.3}+\dots\dots=2e.$

$$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\dots\dots$$

$$e=\quad 1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\dots\dots \qquad \text{(cvii. 2.)}$$

add, $$2e=1+2+\left(1+\frac{1}{\underline{|2}}\right)+\left(\frac{1}{\underline{|3}}+\frac{1}{\underline{|2}}\right)+\left(\frac{1}{\underline{|4}}+\frac{1}{\underline{|3}}\right)+\dots\dots$$

$$=1+2+\frac{3}{\underline{|2}}+\frac{4}{\underline{|3}}+\frac{5}{\underline{|4}}+\dots\dots$$

Ex. 3. Prove that $\log\frac{3}{2}=\frac{1}{2}-\left\{\frac{2}{1.3.2^3}+\frac{3}{2.5.2^5}+\dots\dots\right\}$

$$\log\frac{3}{2}=\log\left(1+\frac{1}{2}\right) \qquad \text{(lxxxix. 1.)}$$

$$=\frac{1}{2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}-\frac{1}{4.2^4}+\frac{1}{5.2^5}-\dots\dots$$

$$=\frac{1}{2}-\frac{3-1}{1.3.2^3}-\frac{5-2}{2.5.2^5}-\dots\dots$$

$$=\frac{1}{2}-\left\{\frac{2}{1.3.2^3}+\frac{3}{2.5.2^5}+\dots\dots\right\}.$$

Ex. 4. Prove $\tan\theta+\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta+\dots=\frac{1}{2}\log\frac{\cos\left(\theta-\frac{\pi}{4}\right)}{\cos\left(\theta+\frac{\pi}{4}\right)}.$

Right-hand member $=\frac{1}{2}\log\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$ (lix. 2.)

$$=\frac{1}{2}\log\frac{1+\tan\theta}{1-\tan\theta}$$

$$=\frac{1}{2}\left\{\log(1+\tan\theta)-\log(1-\tan\theta)\right\}$$

$$=\frac{1}{2}\left\{\tan\theta+\frac{\tan^2\theta}{2}+\frac{\tan^3\theta}{3}+\frac{\tan^4\theta}{4}+\dots\right.$$
$$\left.-\left(-\tan\theta+\frac{\tan^2\theta}{2}-\frac{\tan^3\theta}{3}+\frac{\tan^4\theta}{4}-\dots\right)\right\}$$

$$=\tan\theta+\frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}+\dots$$

13. **Prop.** Find the limit of $\left\{\dfrac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right\}^n$, when $n=\infty$,

$\sin\frac{\theta}{n}$, $\frac{\theta}{n}$, $\tan\frac{\theta}{n}$ are in ascending order of magnitude.

$$\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}} < 1, \text{ and } > \frac{\sin\frac{\theta}{n}}{\tan\frac{\theta}{n}}$$

$$< \quad 1, \quad > \cos\frac{\theta}{n}$$

$$\left\{\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right\}^n < \quad 1^n, \quad > \left\{\cos\frac{\theta}{n}\right\}^n$$

but the limit of $\left(\cos\frac{\theta}{n}\right)^n$ is 1.

∴ the limit of $\left\{\dfrac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right\}^n$ is 1.

Examples.

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Left hand member $=\frac{1+N}{2}\log(1+N)+\frac{1-N}{2}\log(1-N)$

$$=\frac{1}{2}\log(1+N)(1-N)+\frac{N}{2}\log\frac{1+N}{1-N} \qquad \text{(cxix. 3.)}$$

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$$=\frac{1}{2}\left\{-N^2-\frac{N^4}{2}-\frac{N^6}{3}-\frac{N^8}{4}\ldots\ldots\right\}$$

$$+\frac{N}{2}\cdot 2\left\{N+\frac{N^3}{3}+\frac{N^5}{5}+\ldots\ldots\right\}$$

$$=\left(N^2-\frac{N^2}{2}\right)+\left(\frac{N^4}{3}-\frac{N^4}{4}\right)+\left(\frac{N^6}{5}-\frac{N^6}{6}\right)+\ldots\ldots$$

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$$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\dots\dots$$

$$e=\quad 1+\ 1\ +\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\dots\dots \qquad \text{(cvii. 2.)}$$

$$\text{add,}\quad 2e=1+2+\left(1+\frac{1}{\underline{|2}}\right)+\left(\frac{1}{\underline{|3}}+\frac{1}{\underline{|2}}\right)+\left(\frac{1}{\underline{|4}}+\frac{1}{\underline{|3}}\right)+\dots\dots$$

$$=1+2+\frac{3}{\underline{|2}}+\frac{4}{\underline{|3}}+\frac{5}{\underline{|4}}+\dots\dots$$

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$$\log\frac{3}{2}=\log\left(1+\frac{1}{2}\right) \qquad \text{(lxxxix. 1.)}$$

$$=\frac{1}{2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}-\frac{1}{4.2^4}+\frac{1}{5.2^5}-\dots\dots$$

$$=\frac{1}{2}-\frac{3-1}{1.3.2^3}-\frac{5-2}{2.5.2^5}-\dots\dots$$

$$=\frac{1}{2}-\left\{\frac{2}{1.3.2^3}+\frac{3}{2.5.2^5}+\dots\dots\right\}$$

Ex. 4. Prove $\tan\theta+\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta+\dots=\frac{1}{2}\log\frac{\cos\left(\theta-\frac{\pi}{4}\right)}{\cos\left(\theta+\frac{\pi}{4}\right)}.$

$$\text{Right-hand member}=\frac{1}{2}\log\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} \qquad \text{(lix. 2.)}$$

$$=\frac{1}{2}\log\frac{1+\tan\theta}{1-\tan\theta}$$

$$=\frac{1}{2}\left\{\log(1+\tan\theta)-\log(1-\tan\theta)\right\}$$

$$=\frac{1}{2}\left\{\tan\theta+\frac{\tan^2\theta}{2}+\frac{\tan^3\theta}{3}+\frac{\tan^4\theta}{4}+\dots\right.$$
$$\left.-\left(-\tan\theta+\frac{\tan^2\theta}{2}-\frac{\tan^3\theta}{3}+\frac{\tan^4\theta}{4}-\dots\right)\right\}$$

$$=\tan\theta+\frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}+\dots$$

Ex. 5. Prove that $\frac{e}{2}=\frac{1}{\underline{|2}}+\frac{1+2}{\underline{|3}}+\frac{1+2+3}{\underline{|4}}+\text{etc.}\ \ldots$

$$\text{General term}=\frac{1+2+\ldots+n}{\underline{|n+1}}=\frac{n(n+1)}{2\,\underline{|n+1}}=\frac{1}{2}\,\frac{1}{\underline{|n-1}}$$

Writing $n=1, 2, 3\ \ldots$, we have

$$\text{Series}=\frac{1}{2}\left\{1+\frac{1}{1}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\right\}$$

$$=\frac{e}{2}.$$

Ex. 6. Prove that $\frac{1}{e}=\frac{2}{\underline{|3}}+\frac{4}{\underline{|5}}+\frac{6}{\underline{|7}}+\text{etc.}\ \ldots$

$$e^{x}=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}+\ldots$$

$$\therefore\ e^{-x}=1-x+\frac{x^2}{\underline{|2}}-\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}-\ldots$$

put $x=1$, e^{-1} *or* $\frac{1}{e}=1-1+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}-\ldots$

$$=\frac{3-1}{\underline{|3}}+\frac{5-1}{\underline{|5}}+\frac{7-1}{\underline{|7}}+\ldots$$

$$=\frac{2}{\underline{|3}}+\frac{4}{\underline{|5}}+\frac{6}{\underline{|7}}+\ldots$$

XXII.

Theory of Proportional Parts.

1. **Prop.** The change of the Logarithm is approximately proportional to the change of the Number.

$$\text{Log}_{10}(n+d)-\log_{10}n=\log_{10}\frac{n+d}{n}=\log_{10}\left(1+\frac{d}{n}\right)$$

$$=\mu\log_{e}\left(1+\frac{d}{n}\right)$$

$$=\mu\left(\frac{d}{n}-\frac{d^2}{2n^2}+\frac{d^3}{3n^3}-\ldots\right)$$

when $\mu=\cdot 43429448\ \ldots$, a quantity $<\frac{1}{2}$.

If n be a whole number containing five digits, *i.e.*, not $< 10^4$, and if d be not > 1,

$$\text{then } \frac{\mu d^2}{2n^2} \text{ is } < \frac{1}{4}\left(\frac{1}{10^8}\right) < \cdot 000\,000\,003\ \ldots$$

and the next term is much less than this;

$\therefore \log(n+d) - \log n = \mu\frac{d}{n}$, correct to seven places of decimals, the number of places to which logarithms are generally calculated.

Hence if the number be changed from n to $n+d$, the corresponding change in the logarithm is approximately $\frac{\mu d}{n}$; which proves the proposition.

2. The theory of Proportional Parts may be shown to hold in the case of the natural trig-ratios, and also of the log-trig-ratios; that is, the change of each of the trig-ratios, and of the log-trig-ratios is approximately proportional to the change of the angle.

The angles are considered positive, and not > 90°.

3. Prop. To prove the rule for the Table of Natural Sines.

$$\sin(\theta+h) - \sin\theta = \sin h\cos\theta - \sin\theta\,(1-\cos h)$$

$$= \sin h\cos\theta\left(1 - \tan\theta\tan\frac{h}{2}\right)$$

if h is the circular measure of a very small angle, $\sin h = h$ nearly,

$$\sin(\theta+h) - \sin\theta = h\cos\theta\left(1 - \tan\theta\tan\frac{h}{2}\right)$$

if θ is not very nearly $=\frac{\pi}{2}$, $\tan\theta$ is not very large, and $\tan\theta\tan\frac{h}{2}$ may be neglected,

$$\therefore \sin(\theta+h) - \sin\theta = h\cos\theta, \text{ approximately.}$$

$$\text{Similarly } \sin(\theta-h) - \sin\theta = -h\cos\theta,$$

which proves the proposition.

Hence the increment of the sine is very nearly proportional to the cosine, when the angle is not very nearly a right angle.

4. **Prop.** To prove the rule for a Table of Natural Cosines.

$$\cos(\theta-h)-\cos\theta=\sin h\sin\theta-\cos\theta\,(1-\cos h)$$

$$=\sin h\sin\theta\left(1-\cot\theta\tan\frac{h}{2}\right)$$

if h is the circular measure of a very small angle, $\sin h=h$ nearly,

$$\cos(\theta-h)-\cos\theta=h\sin\theta\left(1-\cot\theta\tan\frac{h}{2}\right)$$

if θ is not very small, $\cot\theta$ is not very large, and $\cot\theta\tan\frac{h}{2}$ may be neglected,

$$\therefore\ \cos(\theta-h)-\cos\theta=h\sin\theta, \text{ approximately.}$$

Similarly $\cos(\theta+h)-\cos\theta=-h\sin\theta$,

which proves the proposition.

Hence the increment of the cosine is very nearly proportional to the sine, when the angle is not very small.

5. **Prop.** To prove the rule for a Table of Natural Tangents.

$$\tan(\theta+h)-\tan\theta=\frac{\sin(\theta+h)}{\cos(\theta+h)}-\frac{\sin\theta}{\cos\theta}=\frac{\sin(\theta+h-\theta)}{\cos(\theta+h)\cos\theta}=\frac{\sin h}{\cos(\theta+h)\cos\theta}$$

$$=\frac{\sin h}{\cos^2\theta\,(\cos h-\sin h\tan\theta)}$$

$$=\frac{\tan h}{\cos^2\theta\,(1-\tan h\tan\theta)}$$

If h is the circular measure of a very small angle, $\tan h=h$ nearly, and if θ is not very nearly $=\frac{\pi}{2}$, $\tan\theta\tan h$ may be neglected,

$$\therefore\ \tan(\theta+h)-\tan\theta=\frac{h}{\cos^2\theta}=h\sec^2\theta, \text{ approximately.}$$

Similarly $\tan(\theta-h)-\tan\theta=-h\sec^2\theta$;

which proves the proposition.

By a similar process, we can show—

$$\cot(\theta-h)-\cot\theta=h\,\text{cosec}^2\,\theta \text{ approximately.}$$

$$\cot(\theta+h)-\cot\theta=-h\,\text{cosec}^2\,\theta.$$

6. **Prop.** To prove the rule for a Table of Natural Secants.

$$\sec(\theta+h)-\sec\theta=\frac{1}{\cos(\theta+h)}-\frac{1}{\cos\theta}=\frac{\cos\theta-\cos(\theta+h)}{\cos\theta\cos(\theta+h)}$$

$$=\frac{\sin h\cos\theta+\cos\theta(1-\cos h)}{\cos^2\theta(\cos h-\sin h\tan\theta)}$$

$$=\frac{\tan h\sin\theta\left(1+\tan\frac{h}{2}\cot\theta\right)}{\cos^2\theta(1-\tan h\tan\theta)}$$

If h is the circular measure of a very small angle, $\tan h=h$ nearly, and if θ is neither very small or very nearly $=\frac{\pi}{2}$, $\tan\theta\tan h$ and $\cot\theta\tan\frac{h}{2}$ may be neglected,

$$\therefore\ \sec(\theta+h)-\sec\theta=\frac{h\sin\theta}{\cos^2\theta}=h\sin\theta\sec^2\theta \text{ approximately.}$$

Similarly $\sec(\theta-h)-\sec\theta=-h\sin\theta\sec^2\theta$,

which proves the proposition.

By a similar process we can show

$$\operatorname{cosec}(\theta+h)-\operatorname{cosec}\theta=h\cos\theta\operatorname{cosec}^2\theta \text{ approximately,}$$
$$\operatorname{cosec}(\theta-h)-\operatorname{cosec}\theta=-h\cos\theta\operatorname{cosec}^2\theta.$$

7. **Prop.** To prove the rule for a Table of L sines.

We have $\sin(\theta+h)=\sin\theta\cos h+\cos\theta\sin h$.

If h is the circular measure of a very small angle, $\cos h=1$, $\sin h=h$ nearly,

$$\sin(\theta+h)=\sin\theta+h\cos\theta$$

$$\frac{\sin(\theta+h)}{\sin\theta}=1+h\cot\theta$$

$$L_{10}\sin(\theta+h)-L^{10}\sin\theta=\log_{10}(1+h\cot\theta)$$
$$=\mu\log_e(1+h\cot\theta)$$
$$=\mu\left(h\cot\theta-\frac{h^2}{2}\cot^2\theta+\ldots\right)$$

$$\therefore\ L\sin(\theta+h)-L\sin\theta=\mu h\cot\theta \text{ approximately.}$$

Similarly $L\sin(\theta-h)-L\sin\theta=-\mu h\cot\theta$,

which proves the proposition.

8. **Prop.** To prove the rule for a Table of L cosines.

$$\cos(\theta - h) = \cos\theta \cos h + \sin\theta \sin h.$$

If h be the circular measure of a very small angle $\cos h = 1$, $\sin h = h$ nearly,

$$\cos(\theta - h) = \cos\theta + h \sin\theta$$

$$\frac{\cos(\theta - h)}{\cos\theta} = 1 + h \tan\theta$$

$$\begin{aligned} \therefore\ L_{10}\cos(\theta - h) - L_{10}\cos\theta &= \log_{10}(1 + h\tan\theta) \\ &= \mu \log_e (1 + h\tan\theta) \\ &= \mu\left(h\tan\theta - \frac{1}{2}h^2\tan^2\theta + \ldots\right) \\ &= \mu h \tan\theta \text{ approximately.} \end{aligned}$$

Similarly $L\cos(\theta + h) - L\cos\theta = -\mu h \tan\theta$,

which proves the proposition.

9. **Prop.** To prove the rule for a Table of L tangents.

By **Prop.** 5. $\tan(\theta + h) = \tan\theta + h\sec^2\theta$ approximately

$$\begin{aligned} \frac{\tan(\theta + h)}{\tan\theta} &= 1 + \frac{h\sec^2\theta}{\tan\theta} \\ &= 1 + \frac{2h}{\sin 2\theta} \\ &= 1 + 2h \operatorname{cosec} 2\theta \end{aligned}$$

$$\begin{aligned} \therefore\ L_{10}\tan(\theta + h) - L_{10}\tan\theta &= \log_{10}(1 + 2h \operatorname{cosec} 2\theta) \\ &= \mu \log_e (1 + 2h \operatorname{cosec} 2\theta) \\ &= \mu\, 2h \operatorname{cosec} 2\theta \text{ approximately.} \end{aligned}$$

Similarly $L\tan(\theta - h) - L\tan\theta = -2\mu h \operatorname{cosec} 2\theta$,

which proves the proposition.

10. When θ is very small, $\cot\theta$ is very large, and $\therefore$ $h^2\cot^2\theta$ cannot be neglected; in this case—

$$\sin(\theta+h)-\sin\theta=\sin h\cos\theta\left(1-\tan\theta\tan\frac{h}{2}\right)$$

h being very small, $\sin h=h$, and $\tan\frac{h}{2}=\frac{h}{2}$ approximately,

$$\therefore \sin(\theta+h)-\sin\theta=h\cos\theta-\frac{h^2}{2}\sin\theta$$

$$\frac{\sin(\theta+h)}{\sin\theta}=1+h\cot\theta-\frac{h^2}{2} \qquad \text{(xxxvi. 9.)}$$

$$\therefore L_{10}\sin(\theta+h)-L_{10}\sin\theta=\mu\log_e\left\{1+\left(h\cot\theta-\frac{h^2}{2}\right)\right\}$$

$$=\mu\left(h\cot\theta-\frac{h^2}{2}\right)-\frac{\mu}{2}\left(h\cot\theta-\frac{h^2}{2}\right)^2+\ldots$$

$$=\mu h\cot\theta-\frac{\mu h^2}{2}\left(1+\cot^2\theta\right)+\ldots$$

$$=\mu h\cot\theta-\frac{\mu h^2}{2}\operatorname{cosec}^2\theta+\ldots$$

If the L sines in the tables are calculated to every 10″, then the greatest value of h is $\frac{\pi}{64800}$, or ·00005, and $\mu=\frac{1}{2}$ nearly,

$$\therefore \text{the greatest error } \frac{\mu h^2}{2}\operatorname{cosec}^2\theta=\frac{6\operatorname{cosec}^2\theta}{10^{10}}.$$

This is sensible in calculations to seven places of decimals if $6\operatorname{cosec}^2\theta$ is as great as 10^3, if $\sin^2\theta$ is as small as ·006, or $\sin\theta$ as small as $\surd$·006; and from the tables we see that sin 4° 30′=·0784 . . . is a little greater than $\surd$·006.

So that the differences of consecutive L sines are *irregular* when the angles are very small.

When θ is very nearly a right angle, $\cot\theta$ is very small, but $\operatorname{cosec}^2\theta$ is not very small, and the differences of consecutive L sines in this case are nearly *insensible* and *irregular*.

11. Hence, when an angle is very small or nearly $=\frac{\pi}{2}$, it is impracticable to use the principle of proportional parts in the case of L sines by means of the ordinary tables.

The following methods have been proposed :—

(i.) Tables of L sines are constructed for every second for the first few degrees, so that the greatest value of h is the circular measure of 1″, and $\therefore \frac{\mu h^2}{2}$ cosec$^2\theta$ is small enough to be neglected, if $\sin\theta$ is as small as $\sqrt{\cdot 00006}$; and from the tables we see that sin 30′ is a little more than $\sqrt{\cdot 00006}$, viz., ·0087.

(ii.) **Maskelyne's Method.**

We know that approximately $\sin\theta=\theta-\frac{\theta^3}{6}$ and $\cos\theta=1-\frac{\theta^2}{2}$,

$$\frac{\sin\theta}{\theta}=1-\frac{\theta^2}{6}=\left(1-\frac{\theta^2}{2}\right)^{\frac{1}{3}}=(\cos\theta)^{\frac{1}{3}} \text{ approximately,}$$

$$\therefore \text{L}\sin\theta-10=\log\theta+\frac{1}{3}(\text{L}\cos\theta-10)$$

which gives L $\sin\theta$, if θ be given. If L $\sin\theta$ be given, we must find from the table θ, and then L $\cos\theta$ approximately, so that θ may be found from $\log\theta=\text{L}\sin\theta-10-\frac{1}{3}(\text{L}\cos\theta-10)$.

If θ be the circular measure of n'', $\theta=n\sin 1''$, CHAP. XX. PROP. 4.

$$\sin\theta\ =n\sin 1''(\cos\theta)^{\frac{1}{3}}$$
$$\text{or }\sin n''=n\sin 1''(\cos n'')^{\frac{1}{3}}$$

$$\therefore \text{L}\sin n''=\log n+\text{L}\sin 1''+\frac{1}{3}(\text{L}\cos n''-10)$$

from which we can find L $\sin n''$ from $\log n$, or n if we know L $\sin n''$, taking an approximimate value of L $\cos n''$ from the tables instead of the real value.

(iii.) **Delambre's Method.**

A table is constructed which gives the value of $\log \frac{\sin\theta}{\theta} + \text{L} \sin 1''$ for every second for the first few degrees.

If θ be the circular measure of n'', $\theta = n \sin 1''$,

$$\log \frac{\sin\theta}{\theta} = \log \frac{\sin n''}{n \sin 1''} = \text{L} \sin n'' - \log n - \text{L} \sin 1''$$

$$\therefore \ \log n = \text{L} \sin n'' - \left(\log \frac{\sin\theta}{\theta} + \text{L} \sin 1''\right).$$

If the angle is known, the table gives the value of the expression in brackets, and $\log n$ can be found from the tables of logs of numbers: thus L sin n'' can be found.

If L sin n'' is given, we can find approximately the value of n, and from the table we have the value of the expression in brackets: thus we can find $\log n$, and so n from the tables of logs of numbers.

12. Exceptional cases, where the principle of Proportional Parts is inapplicable.

The differences between the natural sines of two angles, which are each nearly right angles, are very small or *insensible*, since $\cos\theta$ is then very small, and therefore $h \cos\theta$ will be very small; also the second term of the series cannot be neglected, so that the increments of the sines are not proportional to the increments of the angle; they are then said to be *irregular*.

For cosines, they are *insensible and irregular* when the angles are very small.

For tangents, they are *irregular* when the angles are nearly right angles, and never *insensible* since $\sec\theta$ is not less than 1.

For cotangents, they are *irregular* when the angles are very small.

For secants, they are *insensible* when the angles are very small, and *irregular* when nearly right angles.

For cosecants, they are *irregular* when the angles are small, and *insensible* when nearly right angles.

In all cases irregularity accompanies insensibility.

The principle of Proportional Parts fails for every log-trig-ratio, when the angles are either very small or very nearly right angles.

For L sines and L cosecants, the differences are *irregular* when the angles are very small, and *insensible* when nearly right angles.

For L tangents and L cotangents, the differences are *irregular* when the angles are very small or nearly right angles.

For L cosines and L secants, the differences are *insensible* when the angles are very small, and *irregular* when nearly right angles.

13. Since the differences of consecutive sines vary approximately as the cosine, and the differences of consecutive cosines vary approximately as the sine, and

when the angle $< \frac{\pi}{4}$, the sine $<$ cosine,

when the angle $> \frac{\pi}{4}$, the sine $>$ cosine,

Therefore for all angles $< \frac{\pi}{4}$ the differences of consecutive sines will be greater than the differences of consecutive cosines, and *vice versâ*.

Hence for angles $< \frac{\pi}{4}$, it is better to calculate them from their sines, and for angles $> \frac{\pi}{4}$, from their cosines.

The same rule applies to L sines and L cosines.

XXIII.

De Moivre's Theorem.

(i.) $\sqrt{-a^2} = a\sqrt{-1}$; $\sqrt{-1}$ is a symbol subject to the laws of indices, so that $(\sqrt{-1})^2 = -1$, $(\sqrt{-1})^3 = -\sqrt{-1}$, $(\sqrt{-1})^4 = 1$.

Every number is of the form $4m$, $4m+1$, $4m+2$, or $4m+3$,

$$\begin{aligned}
(\sqrt{-1})^{4m} &= (-1)^{2m} = +1\\
(\sqrt{-1})^{4m+1} &= (\sqrt{-1})^{4m}\sqrt{-1} = \sqrt{-1}\\
(\sqrt{-1})^{4m+2} &= (\sqrt{-1})^{4m+1}\sqrt{-1} = \sqrt{-1}\times\sqrt{-1} = -1\\
(\sqrt{-1})^{4m+3} &= (\sqrt{-1})^{4m+2}\sqrt{-1} = -\sqrt{-1}.
\end{aligned}$$

(ii.) If $x + y\sqrt{-1} = a + b\sqrt{-1}$, we know that $x = a$, $y = b$, and that $x - y\sqrt{-1} = a - b\sqrt{-1}$.

(iii.) In the Binomial Theorem, the number of terms in the expansion of $(x+a)^n$ is $n+1$, and if n is even, the coefficient of the middle term $\left(\text{the } \frac{n}{2}+1^{\text{th}}\right)$ is ${}^nC_{\frac{n}{2}}$, or $\dfrac{\underline{|n}}{\underline{|\frac{n}{2}}\,\underline{|\frac{n}{2}}}$: if n is odd, the two middle terms are the $\frac{n+1}{2}$ and the $\frac{n+3}{2}$, and the coefficients are ${}^nC_{\frac{n-1}{2}}$, or $\dfrac{\underline{|n}}{\underline{|\frac{n-1}{2}}\,\underline{|\frac{n+1}{2}}}$.

1. PROP. **De Moivre's Theorem.**

$(\cos\theta \pm \sqrt{-1}\sin\theta)^n = \cos n\theta \pm \sqrt{-1}\sin n\theta$, whatever be the value of n, positive or negative, integral or fractional.

$$\begin{aligned}
(\cos\theta \pm \sqrt{-1}\sin\theta)^2 &= \cos^2\theta - \sin^2\theta \pm 2\sqrt{-1}\sin\theta\cos\theta\\
&= \cos 2\theta \pm \sqrt{-1}\sin 2\theta.
\end{aligned}$$

$$\begin{aligned}
(\cos\theta \pm \sqrt{-1}\sin\theta)^3 &= (\cos 2\theta \pm \sqrt{-1}\sin 2\theta)(\cos\theta \pm \sqrt{-1}\sin\theta)\\
&= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta\\
&\qquad \pm\sqrt{-1}(\sin 2\theta\cos\theta + \cos 2\theta\sin\theta)\\
&= \cos 3\theta \pm \sqrt{-1}\sin 3\theta.\\
&\qquad \text{etc.} \qquad\qquad \text{etc.}
\end{aligned}$$

Assume the law here observed to be true for the index $n-1$; *i.e.* assume

$$(\cos\theta\pm\sqrt{-1}\sin\theta)^{n-1}=\cos(n-1)\theta\pm\sqrt{-1}\sin(n-1)\theta,$$

Introduce a new factor $(\cos\theta\pm\sqrt{-1}\sin\theta)$,

$$\begin{aligned}(\cos\theta\pm\sqrt{-1}\sin\theta)^n&\\ &=\{\cos(n-1)\theta\pm\sqrt{-1}\sin(n-1)\theta\}(\cos\theta\pm\sqrt{-1}\sin\theta)\\ &=\cos(n-1)\theta\cos\theta-\sin(n-1)\theta\sin\theta\\ &\qquad\pm\sqrt{-1}\{\sin(n-1)\theta\cos\theta+\cos(n-1)\theta\sin\theta\}\\ &=\cos n\theta\pm\sqrt{-1}\sin n\theta.\end{aligned}$$

If then the formulæ be true for one value $(n-1)$ of the index, it is proved true for the next (n) value; but it has been shown to be true when the index is 2, 3, . . . and therefore it is true when the index is 4, 5, . . . *i.e.* generally

$$\therefore\ (\cos\theta\pm\sqrt{-1}\sin\theta)^n=\cos n\theta\pm\sqrt{-1}\sin n\theta$$

when n is a positive integer

(ii.) Let $n=-m$.

$$\begin{aligned}\text{Then } (\cos\theta\pm\sqrt{-1}\sin\theta)^n&=(\cos\theta\pm\sqrt{-1}\sin\theta)^{-m}\\ &=\frac{1}{(\cos\theta\pm\sqrt{-1}\sin\theta)^m}\\ &=\frac{1}{\cos m\theta\pm\sqrt{-1}\sin m\theta}\\ &=\frac{\cos m\theta\mp\sqrt{-1}\sin m\theta}{\cos^2 m\theta+\sin^2 m\theta}\\ &=\cos m\theta\mp\sqrt{-1}\sin m\theta\\ &=\cos(-m\theta)\pm\sqrt{-1}\sin(-m\theta)\\ &=\cos n\theta\pm\sqrt{-1}\sin n\theta.\end{aligned}$$

which proves the theorem for a negative integer.

(iii.) Let $n=\frac{p}{q}$, where p and q are integral,

$$(\cos\theta\pm\sqrt{-1}\sin\theta)^n=(\cos\theta\pm\sqrt{-1}\sin\theta)^{\frac{p}{q}}$$

$$=(\cos p\theta\pm\sqrt{-1}\sin p\theta)^{\frac{1}{q}}$$

Now $\qquad (\cos\theta\pm\sqrt{-1}\sin\theta)^n=\cos n\theta\pm\sqrt{-1}\sin n\theta,$
take nth root of both sides, and

$$\cos\theta\pm\sqrt{-1}\sin\theta=(\cos n\theta\pm\sqrt{-1}\sin n\theta)^{\frac{1}{n}}$$

But there are n values of $(\cos n\theta\pm\sqrt{-1}\sin n\theta)^{\frac{1}{n}}$ and one of them is shown to be $\cos\theta\pm\sqrt{-1}\sin\theta$,

$\therefore$ one of the values of $(\cos p\theta\pm\sqrt{-1}\sin p\theta)^{\frac{1}{q}}$ is

$$\cos\frac{p\theta}{q}\pm\sqrt{-1}\sin\frac{p\theta}{q}.$$

It only remains to find an expression which will give *all* the q values of $(\cos\theta\pm\sqrt{-1}\sin\theta)^{\frac{p}{q}}$.

$$(\cos\theta\pm\sqrt{-1}\sin\theta)^{\frac{p}{q}}=(\cos p\theta\pm\sqrt{-1}\sin p\theta)^{\frac{1}{q}}$$

$$=(\cos(2r\pi+p\theta)\pm\sqrt{-1}\sin(2r\pi+p\theta))^{\frac{1}{q}}$$

$$=\cos\frac{2r\pi+p\theta}{q}\pm\sqrt{-1}\sin\frac{2r\pi+p\theta}{q},$$

the q values being found by making $r=0, 1, 2, \ldots, q-1$; and it will be found that there are *only* q different values, for when $r=q$, we have the same value as when $r=0$, and when $r=q+1$, the same value as when $r=1$, etc.

Hence De Moivre's Theorem is fully established.

2. PROP. Find the number of different values of $\cos\frac{2r\pi+p\theta}{q}$ and $\sin\frac{2r\pi+p\theta}{q}$ when successive integral values are assigned to r.

$$r=0,\ \cos\frac{2r\pi+p\theta}{q}=\cos\frac{p\theta}{q}$$

$$r=1,\quad ,, \quad =\cos\frac{2\pi+p\theta}{q} \qquad \text{(lxx. 3.)}$$

$$r=2,\quad ,, \quad =\cos\frac{4\pi+p\theta}{q}$$

etc. etc.

$$r=q-1,\quad ,, \quad =\cos\frac{2(q-1)\pi+p\theta}{q}=\cos\left(2\pi-\frac{2\pi-p\theta}{q}\right)=\cos\frac{2\pi-p\theta}{q}$$

all these q values are *different.*

$$r=q \text{ gives } \cos\frac{2r\pi+p\theta}{q}=\cos\frac{2q\pi+p\theta}{q}=\cos\left(2\pi+\frac{p\theta}{q}\right)=\cos\frac{p\theta}{q},$$

the same value as when $r=0$,

$$r=q+1 \text{ gives } \cos\frac{(2q+2)\pi+p\theta}{q}=\cos\left(2\pi+\frac{2\pi+p\theta}{q}\right)=\cos\frac{2\pi+p\theta}{q},$$

the same value as when $r=1$,

etc.,

from which it appears that there are q and *only* q different values of $\cos\frac{2r\pi+p\theta}{q}$, since the same values afterwards recur in the same order.

Similarly for $\sin\frac{2r\pi+p\theta}{q}$.

3. **Prop.** Express $\sin n\theta$ and $\cos n\theta$ in terms of the powers of $\sin\theta$ and $\cos\theta$.

$$
\begin{aligned}
\cos n\theta + \sqrt{-1}\sin n\theta &= (\cos\theta + \sqrt{-1}\sin\theta)^n \\
&= \cos^n\theta + n\cos^{n-1}\theta\sqrt{-1}\sin\theta + {}^nC_2\cos^{n-2}\theta(\sqrt{-1}\sin\theta)^2 + \dots \\
&\qquad + {}^nC_r\cos^{n-r}\theta(\sqrt{-1}\sin\theta)^r + \dots \\
&\qquad\qquad + n\cos\theta(\sqrt{-1}\sin\theta)^{n-1} + (\sqrt{-1}\sin\theta)^n \\
&= \cos^n\theta + n\cos^{n-1}\theta\sqrt{-1}\sin\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \dots \\
&\qquad\qquad + \dots + (-1)^{\frac{n-1}{2}} n\cos\theta\sin^{n-1}\theta + (-1)^{\frac{n}{2}}\sin^n\theta.
\end{aligned}
$$

Equate the possible and impossible parts,

$$
\therefore \cos n\theta = \cos^n\theta - \frac{n(n-1)}{1.2}\cos^{n-2}\theta\sin^2\theta + \dots \left\{
\begin{array}{l}
+(-1)^{\frac{n}{2}}\sin^n\theta \\
\quad (n \text{ even}) \\
+(-1)^{\frac{n-1}{2}} n\cos\theta\sin^{n-1}\theta \\
\quad (n \text{ odd})
\end{array}\right.
$$

$$
\text{and } \sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\theta\sin^3\theta + \dots
$$

$$
\dots\left\{
\begin{array}{ll}
+(-1)^{\frac{n-2}{2}} n\cos\theta\sin^{n-1}\theta & (n \text{ even}) \\
+(-1)^{\frac{n-1}{2}}\sin^n\theta & (n \text{ odd}).
\end{array}\right.
$$

4. **Prop.** Express $\tan n\theta$ in terms of the powers of $\tan\theta$.

By the previous proposition,

$$
\frac{\sin n\theta}{\cos^n\theta} = n\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - \dots \left\{
\begin{array}{ll}
+(-1)^{\frac{n-2}{2}} n\tan^{n-1}\theta & (n \text{ even}) \\
+(-1)^{\frac{n-1}{2}}\tan^n\theta & (n \text{ odd}).
\end{array}\right.
$$

$$
\frac{\cos n\theta}{\cos^n\theta} = 1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \dots\dots \left\{
\begin{array}{ll}
+(-1)^{\frac{n}{2}}\tan^n\theta & (n \text{ even}) \\
+(-1)^{\frac{n-1}{2}} n\tan^{n-1}\theta & (n \text{ odd}).
\end{array}\right.
$$

By division,

$$
\therefore \tan n\theta = \frac{n\tan\theta - \dfrac{n(n-1)(n-2)}{1.2.3}\tan^3\theta + \dots}{1 - \dfrac{n(n-1)}{1.2}\tan^2\theta + \dots},
$$

writing down the last terms as above.

5. Prop. Express the Sine, Cosine, and Tangent of the sum of any number of angles in terms of the powers of the tangent.

It may be shown that—

$$(\cos a+\sqrt{-1}\,\sin a)\,(\cos\beta+\sqrt{-1}\,\sin\beta)\,(\cos\gamma+\sqrt{-1}\,\sin\gamma)\,\ldots$$
$$=\cos(a+\beta+\gamma+\ldots)+\sqrt{-1}\,\sin(a+\beta+\gamma+\ldots)$$
$$\therefore\ \cos a\cos\beta\cos\gamma\ldots(1+\sqrt{-1}\,\tan a)(1+\sqrt{-1}\,\tan\beta)(1+\sqrt{-1}\,\tan\gamma)\ldots$$
$$=\cos(a+\beta+\gamma+\ldots)+\sqrt{-1}\,\sin(a+\beta+\gamma+\ldots)$$

$$\left.\begin{aligned}\text{let } s_1&=\tan a+\tan\beta+\tan\gamma+\ldots\\ s_2&=\tan a\tan\beta+\tan a\tan\gamma+\ldots\\ s_3&=\tan a\tan\beta\tan\gamma+\ldots\end{aligned}\right\}\text{etc.}\ldots$$

Then $\cos(a+\beta+\gamma+\ldots)+\sqrt{-1}\,\sin(a+\beta+\gamma+\ldots)$
$$=\cos a\cos\beta\cos\gamma\ldots(1-s_1\sqrt{-1}-s_2-s_3\sqrt{-1}+s_4+s_5\sqrt{-1}-\ldots)$$

Equate possible and impossible parts,

$$\therefore\ \sin(a+\beta+\gamma+\ldots)=\cos a\cos\beta\cos\gamma\ldots(s_1-s_3+s_5-s_7\ldots\left\{\begin{matrix}+(-1)^{\frac{n-2}{2}}s_{n-1}\ \binom{n}{\text{even}})\\ +(-1)^{\frac{n-1}{2}}s_n\quad\binom{n}{\text{odd}})\end{matrix}\right.$$

$$\cos(a+\beta+\gamma+\ldots)=\cos a\cos\beta\cos\gamma\ldots(1-s_2+s_4-s_6\ldots\left\{\begin{matrix}+(-1)^{\frac{n}{2}}s_n\quad\binom{n}{\text{even}})\\ +(-1)^{\frac{n-1}{2}}s_{n-1}\ \binom{n}{\text{odd}})\end{matrix}\right.$$

By division,

$$\tan(a+\beta+\gamma+\ldots)=\frac{s_1-s_3+s_5-s_7+\ldots\left.\begin{matrix}\ldots(-1)^{\frac{n-2}{2}}s_{n-1}\ \binom{n}{\text{even}}\\ (-1)^{\frac{n-1}{2}}s_n\quad\binom{n}{\text{odd}}\end{matrix}\right\}}{1-s_2+s_4-s_6+\ldots\left.\begin{matrix}(-1)^{\frac{n}{2}}s_n\quad\binom{n}{\text{even}}\\ \ldots(-1)^{\frac{n-1}{2}}s_{n-1}\ \binom{n}{\text{odd}}\end{matrix}\right\}}$$

Ex. If $\tan A=\frac{1}{2}$, $\tan B=1$, $\tan C=2$, find $\tan(A+B+C)$.

$$\tan(A+B+C)=\frac{s_1-s_3}{1-s_2}=\frac{\left(\frac{1}{2}+1+2\right)-\left(\frac{1}{2}\times1\times2\right)}{1-\frac{1}{2}-1-2}\qquad\text{(cix. 5.)}$$

$$=\frac{\frac{5}{2}}{-\frac{5}{2}}=-1.$$

6. Prop. Expand $\sin a$ and $\cos a$ in a series of powers of a.

When n is a positive integer,

$$\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - \dots$$

$$\sin n\theta = n \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \dots$$

$$\therefore\ \cos n\theta = \cos^n\theta\,(1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta - \dots)$$

$$\sin n\theta = \cos^n\theta\,(n \tan\theta - {}^nC_3 \tan^3\theta + \dots)$$

Let $n\theta = a$, then $n = \frac{a}{\theta}$,

$$\cos a = \cos^n\theta \left\{ 1 - \frac{a(a-\theta)}{\underline{|2}} \left(\frac{\tan\theta}{\theta}\right)^2 + \frac{a(a-\theta)(a-2\theta)(a-3\theta)}{\underline{|4}} \left(\frac{\tan\theta}{\theta}\right)^4 \dots \right\}$$

$$\sin a = \cos^n\theta \left\{ a\,\frac{\tan\theta}{\theta} - \frac{a(a-\theta)(a-2\theta)}{\underline{|3}} \left(\frac{\tan\theta}{\theta}\right)^3 + \dots \right\}$$

if $n = \infty$, then $\theta = 0$, and the limit of $\cos^n\theta$ is 1; also the limit of $\left(\frac{\tan\theta}{\theta}\right)$ and its powers is 1,

$$\therefore\ \cos a = 1 - \frac{a^2}{\underline{|2}} + \frac{a^4}{\underline{|4}} - \dots, \qquad \text{and} \quad \sin a = a - \frac{a^3}{\underline{|3}} + \frac{a^5}{\underline{|5}} - \dots$$

By division, we should get $\tan a = a + \frac{a^3}{3} + \frac{2a^5}{3.5} + \frac{17a^7}{3.3.5.7} + \dots$

(i.) Since $\sin(-a) = -\sin a$, the values of $\sin a$ change sign with a, hence the series for $\sin a$ contains only *odd* powers of a.

And since $\cos(-a) = \cos a$, the series for $\cos a$ contains only *even* powers of a.

(ii.) The series for $\sin a$ and $\cos a$ may be proved to be convergent.

(iii.) $\sin a = a - \left(\frac{a^3}{\underline{|3}} - \frac{a^5}{\underline{|5}}\right) - \left(\frac{a^7}{\underline{|7}} - \frac{a^9}{\underline{|9}}\right) - \dots$

$$= \left(a - \frac{a^3}{\underline{|3}}\right) + \left(\frac{a^5}{\underline{|5}} - \frac{a^7}{\underline{|7}}\right) + \dots$$

The terms in brackets are all + if a^2 is not > 6; *à fortiori*, if a is < 1,

$$\therefore\ \sin a < a \text{ and } > a - \frac{a^3}{6}.\ \text{(As proved in Chap. xx. Prop. 8.)}$$

$$\cos a = 1 - \left(\frac{a^2}{\underline{|2}} - \frac{a^4}{\underline{|4}}\right) - \left(\frac{a^6}{\underline{|6}} - \frac{a^8}{\underline{|8}}\right) - \dots$$

$$= \left(1 - \frac{a^2}{\underline{|2}}\right) + \left(\frac{a^4}{\underline{|4}} - \frac{a^6}{\underline{|6}}\right) + \dots$$

$$\therefore\ \cos a > 1 - \frac{a^2}{2}.$$

7. **Prop.** Expand $\sin n^\circ$ in a series of powers of a, where a is the circular measure of n°.

$$a = \frac{n\pi}{180}$$

$$\therefore \sin n^\circ = \sin a = a - \frac{a^3}{\underline{|3}} + \frac{a^5}{\underline{|5}} - \dots$$

$$= \frac{n\pi}{180} - \frac{1}{\underline{|3}}\left(\frac{n\pi}{180}\right)^3 + \frac{1}{\underline{|5}}\left(\frac{n\pi}{180}\right)^5 - \dots$$

$$\cos n^\circ = \cos a = 1 - \frac{a^2}{\underline{|2}} + \frac{a^4}{\underline{|4}} - \dots$$

$$= 1 - \frac{1}{\underline{|2}}\left(\frac{n\pi}{180}\right)^2 + \frac{1}{\underline{|4}}\left(\frac{n\pi}{180}\right)^4 - \dots$$

EXAMPLES.

Ex. 1. Find the cube roots of 1; *i.e.*, solve $x^3 - 1 = 0$.

$$\sqrt[3]{1} = (\cos 0^\circ + \sqrt{-1}\sin 0^\circ)^{\frac{1}{3}}$$

$$= \cos\frac{2r\pi}{3} + \sqrt{-1}\sin\frac{2r\pi}{3},$$

$$= \cos 0^\circ + \sqrt{-1}\sin 0^\circ = 1, \qquad \text{if } r = 0$$

$$= \cos\frac{2\pi}{3} + \sqrt{-1}\sin\frac{2\pi}{3} = \frac{-1+\sqrt{-3}}{2}, \qquad \text{if } r = 1$$

$$= \cos\frac{4\pi}{3} + \sqrt{-1}\sin\frac{4\pi}{3} = \frac{-1-\sqrt{-3}}{2}, \qquad \text{if } r = 2.$$

Ex. 2. Find the cube roots of -1; *i.e.*, solve $x^3 + 1 = 0$.

$$\sqrt[3]{-1} = (\cos\pi + \sqrt{-1}\sin\pi)^{\frac{1}{3}}$$

$$= \cos\frac{2r\pi+\pi}{3} + \sqrt{-1}\sin\frac{2r\pi+\pi}{3}, \qquad \text{(lxxx. 5.)}$$

$$= \cos\frac{\pi}{3} + \sqrt{-1}\sin\frac{\pi}{3} = \frac{1+\sqrt{-3}}{2}, \qquad \text{if } r = 0$$

$$= \cos\pi + \sqrt{-1}\sin\pi = -1, \qquad \text{if } r = 1$$

$$= \cos\frac{5\pi}{3} + \sqrt{-1}\sin\frac{5\pi}{3} = \frac{1-\sqrt{-3}}{2}, \qquad \text{if } r = 2.$$

Ex. 3. Express the cube roots of $3+4\sqrt{-1}$ in the form $\alpha+\beta\sqrt{-1}$.

$$3+4\sqrt{-1}=5\left(\frac{3}{5}+\frac{4}{5}\sqrt{-1}\right)$$

if $\cos\theta=\dfrac{3}{5}$, we know that $\sin\theta=\dfrac{4}{5}$

$$\therefore\ \sqrt[3]{3+4\sqrt{-1}}=\sqrt[3]{5}\,\{\cos\theta+\sqrt{-1}\sin\theta\}^{\frac{1}{3}}$$
$$=\sqrt[3]{5}\left\{\cos\frac{2r\pi+\theta}{3}+\sqrt{-1}\sin\frac{2r\pi+\theta}{3}\right\}.$$

Where $r=0, 1, 2$, and the cube roots are

$$\sqrt[3]{5}\left\{\cos\frac{\theta}{3}+\sqrt{-1}\sin\frac{\theta}{3}\right\},\quad \sqrt[3]{5}\left\{\cos\frac{2\pi+\theta}{3}+\sqrt{-1}\sin\frac{2\pi+\theta}{3}\right\},$$
$$\sqrt[3]{5}\left\{\cos\frac{4\pi+\theta}{3}+\sqrt{-1}\sin\frac{4\pi+\theta}{3}\right\}.$$

And from a table of natural cosines,

$\cos\theta=\cdot 6$ gives $\theta=53°\,8'$ nearly.

Ex. 4. Find the value of $(-1+\sqrt{-3})^5+(-1-\sqrt{-3})^5$. (lxxiii. 1.)

$$\text{expression }=2^5\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1}\right)^5+2^5\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\sqrt{-1}\right)^5$$
$$=2^5\left\{\left(\cos\frac{2\pi}{3}+\sqrt{-1}\sin\frac{2\pi}{3}\right)^5+\left(\cos\frac{2\pi}{3}-\sqrt{-1}\sin\frac{2\pi}{3}\right)^5\right\}$$
$$=2^5\left\{\cos\frac{10\pi}{3}+\sqrt{-1}\sin\frac{10\pi}{3}+\cos\frac{10\pi}{3}-\sqrt{-1}\sin\frac{10\pi}{3}\right\}$$
$$=2^6\cos\frac{10\pi}{3}=2^6\times\frac{1}{2}=2^5=32.$$

Ex. 5. If $\tan\theta=\dfrac{b}{a}$, prove that

$$(a\pm b\sqrt{-1})^{\frac{1}{n}}=(a^2+b^2)^{\frac{1}{2n}}\left(\cos\frac{\theta}{n}\pm\sqrt{-1}\sin\frac{\theta}{n}\right).$$

$\tan\theta=\dfrac{b}{a}$, $\therefore\sin\theta=\dfrac{b}{\sqrt{a^2+b^2}}$ and $\cos\theta=\dfrac{a}{\sqrt{a^2+b^2}}$, (cii. 7.)

$$\therefore\ (a\pm b\sqrt{-1})^{\frac{1}{n}}=\left\{\frac{a}{\sqrt{a^2+b^2}}\pm\frac{b}{\sqrt{a^2+b^2}}\sqrt{-1}\right\}^{\frac{1}{n}}(a^2+b^2)^{\frac{1}{2n}}$$
$$=(\cos\theta\pm\sqrt{-1}\sin\theta)^{\frac{1}{n}}(a^2+b^2)^{\frac{1}{2n}}$$
$$=\left(\cos\frac{\theta}{n}\pm\sqrt{-1}\sin\frac{\theta}{n}\right)(a^2+b^2)^{\frac{1}{2n}}.$$

Ex. 6. Express $\sqrt[6]{-1}$ in the form $a+\beta\sqrt{-1}$; determine all the values of a and β.

$$\sqrt[6]{-1}=(\cos\pi\pm\sqrt{-1}\sin\pi)^{\frac{1}{6}} \qquad \text{(cxvi. 6.)}$$

$$=\cos\frac{(2r+1)\pi}{6}\pm\sqrt{-1}\sin\frac{(2r+1)\pi}{6},$$

$$=\cos\frac{\pi}{6}\pm\sqrt{-1}\sin\frac{\pi}{6}=\frac{\sqrt{3}\pm\sqrt{-1}}{2}, \qquad \text{if } r=0$$

$$=\cos\frac{\pi}{2}\pm\sqrt{-1}\sin\frac{\pi}{2}=\pm\sqrt{-1}, \qquad \text{if } r=1$$

$$=\cos\frac{5\pi}{6}\pm\sqrt{-1}\sin\frac{5\pi}{6}=\frac{-\sqrt{3}\pm\sqrt{-1}}{2}, \qquad \text{if } r=2.$$

The six values are $\pm\sqrt{-1}$ and $\frac{\pm\sqrt{3}\pm\sqrt{-1}}{2}$.

XXIV.

Expansion of Trigonometrical Functions.

1. PROP. If $x=\cos\theta+\sqrt{-1}\sin\theta$,

$$\frac{1}{x}=\frac{1}{\cos\theta+\sqrt{-1}\sin\theta}=\frac{\cos\theta-\sqrt{-1}\sin\theta}{\cos^2\theta+\sin^2\theta}=\cos\theta-\sqrt{-1}\sin\theta,$$

$$\therefore x+\frac{1}{x}=2\cos\theta, \qquad x-\frac{1}{x}=2\sqrt{-1}\sin\theta.$$

Also $x^n=(\cos\theta+\sqrt{-1}\sin\theta)^n=\cos n\theta+\sqrt{-1}\sin n\theta$,

$$\frac{1}{x^n}=(\cos\theta-\sqrt{-1}\sin\theta)^n=\cos n\theta-\sqrt{-1}\sin n\theta,$$

$$\therefore x^n+\frac{1}{x^n}=2\cos n\theta, \qquad x^n-\frac{1}{x^n}=2\sqrt{-1}\sin n\theta.$$

If $x=\cos\theta+\sqrt{-1}\sin\theta$, $\therefore \frac{1}{x}=\cos\theta-\sqrt{-1}\sin\theta$,

$y=\cos\phi+\sqrt{-1}\sin\phi$, $\therefore \frac{1}{y}=\cos\phi-\sqrt{-1}\sin\phi$,

$$\therefore xy+\frac{1}{xy}=2\cos(\theta+\phi), \qquad xy-\frac{1}{xy}=2\sqrt{-1}\sin(\theta+\phi).$$

2. **Prop.** Express $\cos^n\theta$ in terms of cosines of multiples of θ, when n is a positive integer.

$$2\cos\theta = x + \frac{1}{x}$$

$$2^n\cos^n\theta = \left(x+\frac{1}{x}\right)^n$$

$$= x^n + nx^{n-2} + {}^nC_2x^{n-4} + \dots \left\{\begin{matrix} +{}^nC_{\frac{n}{2}} & (n \text{ even}) \\ +{}^nC_{\frac{n-1}{2}}\left(x+\frac{1}{x}\right) & (n \text{ odd}) \end{matrix}\right\} + \dots$$

$$\dots + {}^nC_2\frac{1}{x^{n-4}} + n\frac{1}{x^{n-2}} + \frac{1}{x^n}$$

$$= \left(x^n+\frac{1}{x^n}\right) + n\left(x^{n-2}+\frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4}+\frac{1}{x^{n-4}}\right) + \dots \left\{\begin{matrix} +{}^nC_{\frac{n}{2}} & (n \text{ even}) \\ +\cdot C_{\frac{n-1}{2}}\left(x+\frac{1}{x}\right) & (n \text{ odd}) \end{matrix}\right\}$$

$$= 2\cos n\theta + n\,2\cos(n-2)\theta + \frac{n(n-1)}{\underline{|2}}\,2\cos(n-4)\theta + \dots$$

$$\left.\begin{matrix} \dots + \dfrac{\underline{|n}}{\underline{|\frac{n}{2}}\,\underline{|\frac{n}{2}}} & (n \text{ even}) \\ \\ \dots + \dfrac{\underline{|n}}{\underline{|\frac{n-1}{2}}\,\underline{|\frac{n+1}{2}}}2\cos\theta & (n \text{ odd}) \end{matrix}\right\}$$

$$\therefore\ 2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta + \dots$$

$$+ \frac{\underline{|n}}{2\,\underline{|\frac{n}{2}}\,\underline{|\frac{n}{2}}} \text{ or } \frac{\underline{|n}}{\underline{|\frac{n-1}{2}}\,\underline{|\frac{n+1}{2}}}\cos\theta.$$

[illegible] of multiples [illegible]

[illegible]

[illegible] multiples of θ [illegible]

[illegible]

5. PROP. Expand $\cos n\theta$ in a series of descending powers of $\cos\theta$, when n is a positive integer.

Let $x=\cos\theta+\sqrt{-1}\sin\theta$

Then $x+\dfrac{1}{x}=2\cos\theta$, and $x^n+\dfrac{1}{x^n}=2\cos n\theta$.

$$\text{Now } (1-zx)\left(1-\frac{z}{x}\right)=1-z\left(x+\frac{1}{x}\right)+z^2$$

$$=1-z\,(2\cos\theta-z)$$

$$=1-z\,(c-z),\quad \text{if } c=2\cos\theta,$$

$$\log(1-zx)+\log\left(1-\frac{z}{x}\right)=\log(1-z\,(c-z))$$

$$\therefore\ zx+\frac{z^2x^2}{2}+\ldots+\frac{z^nx^n}{n}+\ldots+\frac{z}{x}+\frac{z^2}{2x^2}+\ldots+\frac{z^n}{nx^n}+\ldots$$

$$=z\,(c-z)+\frac{z^2\,(c-z)^2}{2}+\ldots+\frac{z^{n-1}\,(c-z)^{n-1}}{n-1}+\frac{z^n\,(c-z)^n}{n}+\ldots$$

Coefficient of z^n on the left-hand side $=\dfrac{1}{n}\left(x^n+\dfrac{1}{x^n}\right)$,

Coefficient of z^n in $\dfrac{z^n\,(c-z)^n}{n}$ is $\dfrac{c^n}{n}$,

in $\dfrac{z^{n-1}\,(c-z)^{n-1}}{n-1}$ is $\dfrac{1}{n-1}\{-(n-1)\,c^{n-2}\}$,

in $\dfrac{z^{n-2}\,(c-z)^{n-2}}{n-2}$ is $\dfrac{1}{n-2}\,\dfrac{(n-2)\,(n-3)}{1.2}\,c^{n-4}$,

etc. etc.

$$\frac{1}{n}\,2\cos n\theta=\frac{1}{n}\,(2\cos\theta)^n-(2\cos\theta)^{n-2}+\frac{n-3}{1.2}\,(2\cos\theta)^{n-4}-\ldots$$

$$\therefore\ \cos n\theta=2^{n-1}\cos^n\theta-n\,2^{n-3}\cos^{n-2}\theta+\frac{n\,(n-3)}{1.2}\,2^{n-5}\cos^{n-4}\theta-\ldots$$

Ex. $\cos 5\theta=2^4\cos^5\theta-5.2^2\cos^3\theta+\dfrac{5.2}{1.2}\cos\theta$

$$=16\cos^5\theta-20\cos^3\theta+5\cos\theta. \qquad \text{(li. 6.)}$$

3. **Prop.** Express $\sin^n\theta$ in terms of cosines of multiples of θ, when n is an even positive integer.

$$2\sqrt{-1}\sin\theta = x - \frac{1}{x}$$

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = x^n - nx^{n-2} + {}^nC_2\,x^{n-4} + \dots$$

$$+(-1)^{\frac{n}{2}}\,{}^nC_{\frac{n}{2}} + \dots + {}^nC_2\frac{1}{x^{n-4}} - n\frac{1}{x^{n-2}} + \frac{1}{x^n}$$

$$=\left(x^n+\frac{1}{x^n}\right) - n\left(x^{n-2}+\frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4}+\frac{1}{x^{n-4}}\right) + \dots + (-1)^{\frac{n}{2}}\,{}^nC_{\frac{n}{2}}$$

$$=2\cos n\theta - n\,2\cos(n-2)\theta + \frac{n(n-1)}{1.2}\,2\cos(n-4)\theta + \dots$$

$$+(-1)^{\frac{n}{2}}\frac{\underline{|n}}{\underline{|\tfrac{n}{2}}\;\underline{|\tfrac{n}{2}}}$$

$$\therefore\ 2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta \dots$$

$$+(-1)^{\frac{n}{2}}\frac{\underline{|n}}{2\,\underline{|\tfrac{n}{2}}\;\underline{|\tfrac{n}{2}}}$$

4. **Prop.** Express $\sin^n\theta$ in terms of sines of multiples of θ when n is an odd positive integer.

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = \left(x-\frac{1}{x}\right)^n = x^n - nx^{n-2} + {}^nC_2\,x^{n-4} \dots$$

$$\dots + (-1)^{\frac{n-1}{2}}\,{}^nC_{\frac{n-1}{2}}\left(x-\frac{1}{x}\right) \dots - {}^nC_2\frac{1}{x^{n-4}} + n\frac{1}{x^{n-2}} - \frac{1}{x^n}$$

$$=\left(x^n-\frac{1}{x^n}\right) - n\left(x^{n-2}-\frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4}-\frac{1}{x^{n-4}}\right) - \dots$$

$$+(-1)^{\frac{n-1}{2}}\,{}^nC_{\frac{n-1}{2}}\left(x-\frac{1}{x}\right)$$

$$=2\sqrt{-1}\sin n\theta - n\,2\sqrt{-1}\sin(n-2)\theta + \frac{n(n-1)}{1.2}\,2\sqrt{-1}\sin(n-4)\theta - \dots$$

$$+(-1)^{\frac{n-1}{2}}\frac{\underline{|n}}{\underline{|\tfrac{n-1}{2}}\;\underline{|\tfrac{n+1}{2}}}\,2\sqrt{-1}\sin\theta$$

$$\therefore\ 2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1.2}\sin(n-4)\theta - \dots$$

$$+(-1)^{\frac{n-1}{2}}\frac{\underline{|n}}{\underline{|\tfrac{n-1}{2}}\;\underline{|\tfrac{n+1}{2}}}\sin\theta.$$

5. **Prop.** Expand $\cos n\theta$ in a series of descending powers of $\cos\theta$, when n is a positive integer.

Let $x=\cos\theta+\sqrt{-1}\sin\theta$

Then $x+\dfrac{1}{x}=2\cos\theta$, and $x^n+\dfrac{1}{x^n}=2\cos n\theta$.

$$\text{Now } (1-zx)\left(1-\frac{z}{x}\right)=1-z\left(x+\frac{1}{x}\right)+z^2$$

$$=1-z\,(2\cos\theta-z)$$

$$=1-z\,(c-z),\quad \text{if } c=2\cos\theta,$$

$$\log\,(1-zx)+\log\left(1-\frac{z}{x}\right)=\log\,(1-z\,(c-z))$$

$$\therefore\ zx+\frac{z^2x^2}{2}+\ldots+\frac{z^nx^n}{n}+\ldots+\frac{z}{x}+\frac{z^2}{2x^2}+\ldots+\frac{z^n}{nx^n}+\ldots$$

$$=z\,(c-z)+\frac{z^2\,(c-z)^2}{2}+\ldots+\frac{z^{n-1}\,(c-z)^{n-1}}{n-1}+\frac{z^n\,(c-z)^n}{n}+\ldots$$

Coefficient of z^n on the left-hand side $=\dfrac{1}{n}\left(x^n+\dfrac{1}{x^n}\right)$,

Coefficient of z^n in $\dfrac{z^n\,(c-z)^n}{n}$ is $\dfrac{c^n}{n}$,

in $\dfrac{z^{n-1}\,(c-z)^{n-1}}{n-1}$ is $\dfrac{1}{n-1}\,\{-(n-1)\,c^{n-2}\}$,

in $\dfrac{z^{n-2}\,(c-z)^{n-2}}{n-2}$ is $\dfrac{1}{n-2}\,\dfrac{(n-2)\,(n-3)}{1.2}\,c^{n-4}$,

etc. etc.

$$\frac{1}{n}\,2\cos n\theta=\frac{1}{n}\,(2\cos\theta)^n-(2\cos\theta)^{n-2}+\frac{n-3}{1.2}\,(2\cos\theta)^{n-4}-\ldots$$

$$\therefore\ \cos n\theta=2^{n-1}\cos^n\theta-n\,2^{n-3}\cos^{n-2}\theta+\frac{n\,(n-3)}{1.2}\,2^{n-5}\cos^{n-4}\theta-\cdots$$

Ex. $\cos 5\theta=2^4\cos^5\theta-5.2^2\cos^3\theta+\dfrac{5.2}{1.2}\cos\theta$

$$=16\cos^5\theta-20\cos^3\theta+5\cos\theta. \qquad \text{(li. 6.)}$$

3. **Prop.** Express $\sin^n\theta$ in terms of cosines of multiples of θ, when n is an even positive integer.

$$2\sqrt{-1}\sin\theta = x - \frac{1}{x}$$

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = x^n - nx^{n-2} + {}^nC_2\,x^{n-4} + \dots$$

$$+ (-1)^{\frac{n}{2}}\,{}^nC_{\frac{n}{2}} + \dots + {}^nC_2\frac{1}{x^{n-4}} - n\frac{1}{x^{n-2}} + \frac{1}{x^n}$$

$$= \left(x^n + \frac{1}{x^n}\right) - n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \dots + (-1)^{\frac{n}{2}}\,{}^nC_{\frac{n}{2}}$$

$$= 2\cos n\theta - n\,2\cos(n-2)\theta + \frac{n(n-1)}{1.2}\,2\cos(n-4)\theta + \dots$$

$$+ (-1)^{\frac{n}{2}}\frac{\underline{|n}}{\underline{|\frac{n}{2}}\,\underline{|\frac{n}{2}}}$$

$$\therefore\ 2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta \dots$$

$$+ (-1)^{\frac{n}{2}}\frac{\underline{|n}}{2\,\underline{|\frac{n}{2}}\,\underline{|\frac{n}{2}}}$$

4. **Prop.** Express $\sin^n\theta$ in terms of sines of multiples of θ when n is an odd positive integer.

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = \left(x - \frac{1}{x}\right)^n = x^n - nx^{n-2} + {}^nC_2\,x^{n-4} \dots$$

$$\dots + (-1)^{\frac{n-1}{2}}\,{}^nC_{\frac{n-1}{2}}\left(x - \frac{1}{x}\right) \dots - {}^nC_2\frac{1}{x^{n-4}} + n\frac{1}{x^{n-2}} - \frac{1}{x^n}$$

$$= \left(x^n - \frac{1}{x^n}\right) - n\left(x^{n-2} - \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} - \frac{1}{x^{n-4}}\right) - \dots$$

$$+ (-1)^{\frac{n-1}{2}}\,{}^nC_{\frac{n-1}{2}}\left(x - \frac{1}{x}\right)$$

$$= 2\sqrt{-1}\sin n\theta - n\,2\sqrt{-1}\sin(n-2)\theta + \frac{n(n-1)}{1.2}\,2\sqrt{-1}\sin(n-4)\theta - \dots$$

$$+ (-1)^{\frac{n-1}{2}}\frac{\underline{|n}}{\underline{|\frac{n-1}{2}}\,\underline{|\frac{n+1}{2}}}\,2\sqrt{-1}\sin\theta$$

$$\therefore\ 2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1.2}\sin(n-4)\theta - \dots$$

$$+ (-1)^{\frac{n-1}{2}}\frac{\underline{|n}}{\underline{|\frac{n-1}{2}}\,\underline{|\frac{n+1}{2}}}\sin\theta.$$

5. Prop. Expand $\cos n\theta$ in a series of descending powers of $\cos\theta$, when n is a positive integer.

$$\text{Let } x=\cos\theta+\sqrt{-1}\sin\theta$$

$$\text{Then } x+\frac{1}{x}=2\cos\theta, \quad \text{and } x^n+\frac{1}{x^n}=2\cos n\theta.$$

$$\text{Now } (1-zx)\left(1-\frac{z}{x}\right)=1-z\left(x+\frac{1}{x}\right)+z^2$$

$$=1-z(2\cos\theta-z)$$

$$=1-z(c-z), \quad \text{if } c=2\cos\theta,$$

$$\log(1-zx)+\log\left(1-\frac{z}{x}\right)=\log(1-z(c-z))$$

$$\therefore\ zx+\frac{z^2x^2}{2}+\ldots+\frac{z^nx^n}{n}+\ldots+\frac{z}{x}+\frac{z^2}{2x^2}+\ldots+\frac{z^n}{nx^n}+\ldots$$

$$=z(c-z)+\frac{z^2(c-z)^2}{2}+\ldots+\frac{z^{n-1}(c-z)^{n-1}}{n-1}+\frac{z^n(c-z)^n}{n}+\ldots$$

Coefficient of z^n on the left-hand side $=\dfrac{1}{n}\left(x^n+\dfrac{1}{x^n}\right)$,

Coefficient of z^n in $\dfrac{z^n(c-z)^n}{n}$ is $\dfrac{c^n}{n}$,

in $\dfrac{z^{n-1}(c-z)^{n-1}}{n-1}$ is $\dfrac{1}{n-1}\{-(n-1)c^{n-2}\}$,

in $\dfrac{z^{n-2}(c-z)^{n-2}}{n-2}$ is $\dfrac{1}{n-2}\,\dfrac{(n-2)(n-3)}{1.2}\,c^{n-4}$,

etc. etc.

$$\frac{1}{n}\,2\cos n\theta=\frac{1}{n}(2\cos\theta)^n-(2\cos\theta)^{n-2}+\frac{n-3}{1.2}(2\cos\theta)^{n-4}-\ldots$$

$$\therefore\ \cos n\theta=2^{n-1}\cos^n\theta-n\,2^{n-3}\cos^{n-2}\theta+\frac{n(n-3)}{1.2}\,2^{n-5}\cos^{n-4}\theta-\ldots$$

Ex. $\cos 5\theta=2^4\cos^5\theta-5.2^2\cos^3\theta+\dfrac{5.2}{1.2}\cos\theta$

$$=16\cos^5\theta-20\cos^3\theta+5\cos\theta. \qquad \text{(li. 6.)}$$

3. **Prop.** Express $\sin^n\theta$ in terms of cosines of multiples of θ, when n is an even positive integer.

$$2\sqrt{-1}\sin\theta = x - \frac{1}{x}$$

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = x^n - nx^{n-2} + {}^nC_2\,x^{n-4} + \dots$$

$$+(-1)^{\frac{n}{2}}\,{}^nC_{\frac{n}{2}} + \dots + {}^nC_2\frac{1}{x^{n-4}} - n\frac{1}{x^{n-2}} + \frac{1}{x^n}$$

$$= \left(x^n + \frac{1}{x^n}\right) - n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \dots + (-1)^{\frac{n}{2}}\,{}^nC_{\frac{n}{2}}$$

$$= 2\cos n\theta - n\,2\cos(n-2)\theta + \frac{n(n-1)}{1.2}\,2\cos(n-4)\theta + \dots$$

$$+(-1)^{\frac{n}{2}}\frac{\underline{|n}}{\underline{|\frac{n}{2}}\,\underline{|\frac{n}{2}}}$$

$$\therefore\ 2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta \dots$$

$$+(-1)^{\frac{n}{2}}\frac{\underline{|n}}{2\,\underline{|\frac{n}{2}}\,\underline{|\frac{n}{2}}}$$

4. **Prop.** Express $\sin^n\theta$ in terms of sines of multiples of θ when n is an odd positive integer.

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = \left(x - \frac{1}{x}\right)^n = x^n - nx^{n-2} + {}^nC_2\,x^{n-4}\dots$$

$$\dots + (-1)^{\frac{n-1}{2}}\,{}^nC_{\frac{n-1}{2}}\left(x - \frac{1}{x}\right)\dots - {}^nC_2\frac{1}{x^{n-4}} + n\frac{1}{x^{n-2}} - \frac{1}{x^n}$$

$$= \left(x^n - \frac{1}{x^n}\right) - n\left(x^{n-2} - \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} - \frac{1}{x^{n-4}}\right) - \dots$$

$$+(-1)^{\frac{n-1}{2}}\,{}^nC_{\frac{n-1}{2}}\left(x - \frac{1}{x}\right)$$

$$= 2\sqrt{-1}\sin n\theta - n\,2\sqrt{-1}\sin(n-2)\theta + \frac{n(n-1)}{1.2}\,2\sqrt{-1}\sin(n-4)\theta - \dots$$

$$+(-1)^{\frac{n-1}{2}}\frac{\underline{|n}}{\underline{|\frac{n-1}{2}}\,\underline{|\frac{n+1}{2}}}\,2\sqrt{-1}\sin\theta$$

$$\therefore\ 2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1.2}\sin(n-4)\theta - \dots$$

$$+(-1)^{\frac{n-1}{2}}\frac{\underline{|n}}{\underline{|\frac{n-1}{2}}\,\underline{|\frac{n+1}{2}}}\sin\theta.$$

5. PROP. Expand $\cos n\theta$ in a series of descending powers of $\cos\theta$, when n is a positive integer.

$$\text{Let } x = \cos\theta + \sqrt{-1}\,\sin\theta$$

$$\text{Then } x + \frac{1}{x} = 2\cos\theta, \quad \text{and } x^n + \frac{1}{x^n} = 2\cos n\theta.$$

$$\text{Now } (1-zx)\left(1-\frac{z}{x}\right) = 1 - z\left(x+\frac{1}{x}\right) + z^2$$

$$= 1 - z\,(2\cos\theta - z)$$

$$= 1 - z\,(c-z), \quad \text{if } c = 2\cos\theta,$$

$$\log(1-zx) + \log\left(1-\frac{z}{x}\right) = \log(1 - z\,(c-z))$$

$$\therefore\ zx + \frac{z^2x^2}{2} + \ldots + \frac{z^nx^n}{n} + \ldots + \frac{z}{x} + \frac{z^2}{2x^2} + \ldots + \frac{z^n}{nx^n} + \ldots$$

$$= z\,(c-z) + \frac{z^2\,(c-z)^2}{2} + \ldots + \frac{z^{n-1}\,(c-z)^{n-1}}{n-1} + \frac{z^n\,(c-z)^n}{n} + \ldots$$

Coefficient of z^n on the left-hand side $= \dfrac{1}{n}\left(x^n + \dfrac{1}{x^n}\right)$,

Coefficient of z^n in $\dfrac{z^n\,(c-z)^n}{n}$ is $\dfrac{c^n}{n}$,

in $\dfrac{z^{n-1}\,(c-z)^{n-1}}{n-1}$ is $\dfrac{1}{n-1}\{-(n-1)\,c^{n-2}\}$,

in $\dfrac{z^{n-2}\,(c-z)^{n-2}}{n-2}$ is $\dfrac{1}{n-2}\,\dfrac{(n-2)\,(n-3)}{1.2}\,c^{n-4}$,

etc. etc.

$$\frac{1}{n}\,2\cos n\theta = \frac{1}{n}\,(2\cos\theta)^n - (2\cos\theta)^{n-2} + \frac{n-3}{1.2}\,(2\cos\theta)^{n-4} - \ldots$$

$$\therefore\ \cos n\theta = 2^{n-1}\cos^n\theta - n\,2^{n-3}\cos^{n-2}\theta + \frac{n\,(n-3)}{1.2}\,2^{n-5}\cos^{n-4}\theta - \cdots$$

Ex. $\cos 5\theta = 2^4\cos^5\theta - 5.2^2\cos^3\theta + \dfrac{5.2}{1.2}\cos\theta$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta. \qquad \text{(li. 6.)}$$

Ex. 6. Express $\sqrt[6]{-1}$ **in the form** $a+\beta\sqrt{-1}$; **determine all the values of** a **and** β.

$$\sqrt[6]{-1}=(\cos\pi\pm\sqrt{-1}\sin\pi)^{\frac{1}{6}} \qquad \text{(cxvi. 6.)}$$

$$=\cos\frac{(2r+1)\pi}{6}\pm\sqrt{-1}\sin\frac{(2r+1)\pi}{6},$$

$$=\cos\frac{\pi}{6}\pm\sqrt{-1}\sin\frac{\pi}{6}=\frac{\sqrt{3}\pm\sqrt{-1}}{2}, \qquad \text{if } r=0$$

$$=\cos\frac{\pi}{2}\pm\sqrt{-1}\sin\frac{\pi}{2}=\pm\sqrt{-1}, \qquad \text{if } r=1$$

$$=\cos\frac{5\pi}{6}\pm\sqrt{-1}\sin\frac{5\pi}{6}=\frac{-\sqrt{3}\pm\sqrt{-1}}{2}, \qquad \text{if } r=2.$$

The six values are $\pm\sqrt{-1}$ and $\dfrac{\pm\sqrt{3}\pm\sqrt{-1}}{2}$.

XXIV.

Expansion of Trigonometrical Functions.

1. Prop. If $x=\cos\theta+\sqrt{-1}\sin\theta$,

$$\frac{1}{x}=\frac{1}{\cos\theta+\sqrt{-1}\sin\theta}=\frac{\cos\theta-\sqrt{-1}\sin\theta}{\cos^2\theta+\sin^2\theta}=\cos\theta-\sqrt{-1}\sin\theta,$$

$$\therefore x+\frac{1}{x}=2\cos\theta, \qquad x-\frac{1}{x}=2\sqrt{-1}\sin\theta.$$

Also $x^n=(\cos\theta+\sqrt{-1}\sin\theta)^n=\cos n\theta+\sqrt{-1}\sin n\theta$,

$$\frac{1}{x^n}=(\cos\theta-\sqrt{-1}\sin\theta)^n=\cos n\theta-\sqrt{-1}\sin n\theta,$$

$$\therefore x^n+\frac{1}{x^n}=2\cos n\theta, \qquad x^n-\frac{1}{x^n}=2\sqrt{-1}\sin n\theta.$$

If $x=\cos\theta+\sqrt{-1}\sin\theta$, $\therefore \dfrac{1}{x}=\cos\theta-\sqrt{-1}\sin\theta$,

$y=\cos\phi+\sqrt{-1}\sin\phi$, $\therefore \dfrac{1}{y}=\cos\phi-\sqrt{-1}\sin\phi$,

$$\therefore xy+\frac{1}{xy}=2\cos(\theta+\phi), \qquad xy-\frac{1}{xy}=2\sqrt{-1}\sin(\theta+\phi).$$

2. Prop. Express $\cos^n\theta$ in terms of cosines of multiples of θ, when n is a positive integer.

$$2\cos\theta = x+\frac{1}{x}$$

$$2^n\cos^n\theta = \left(x+\frac{1}{x}\right)^n$$

$$= x^n + nx^{n-2} + {}^nC_2x^{n-4} + \dots \left\{\begin{array}{ll} +{}^nC_{\frac{n}{2}} & (n \text{ even}) \\ +{}^nC_{\frac{n-1}{2}}\left(x+\frac{1}{x}\right) & (n \text{ odd}) \end{array}\right\} + \dots$$

$$\dots + {}^nC_2\frac{1}{x^{n-4}} + n\frac{1}{x^{n-2}} + \frac{1}{x^n}$$

$$= \left(x^n+\frac{1}{x^n}\right) + n\left(x^{n-2}+\frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4}+\frac{1}{x^{n-4}}\right) + \dots \left\{\begin{array}{ll} +{}^nC_{\frac{n}{2}} & \left(\substack{n \\ \text{even}}\right) \\ +\cdot C_{\frac{n-1}{2}}\left(x+\frac{1}{x}\right) & \left(\substack{n \\ \text{odd}}\right) \end{array}\right\}$$

$$= 2\cos n\theta + n\;2\cos(n-2)\theta + \frac{n(n-1)}{\underline{|2}}\,2\cos(n-4)\theta + \dots$$

$$\left.\begin{array}{ll} \dots + \dfrac{\underline{|n}}{\underline{|\frac{n}{2}}\,\underline{|\frac{n}{2}}} & (n \text{ even}) \\ \dots + \dfrac{\underline{|n}}{\underline{|\frac{n-1}{2}}\,\underline{|\frac{n+1}{2}}}\,2\cos\theta & (n \text{ odd}) \end{array}\right\}$$

$$\therefore\; 2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta + \dots$$

$$+\frac{\underline{|n}}{2\,\underline{|\frac{n}{2}}\,\underline{|\frac{n}{2}}} \text{ or } \frac{\underline{|n}}{\underline{|\frac{n-1}{2}}\,\underline{|\frac{n+1}{2}}}\cos\theta.$$

3. Prop. Express θ in terms of Powers of $\tan\theta$.

Gregory's Series.

$$e^{\theta\sqrt{-1}}=\cos\theta+\sqrt{-1}\sin\theta$$
$$e^{-\theta\sqrt{-1}}=\cos\theta-\sqrt{-1}\sin\theta$$

By division, $e^{2\theta\sqrt{-1}}=\dfrac{\cos\theta+\sqrt{-1}\sin\theta}{\cos\theta-\sqrt{-1}\sin\theta}=\dfrac{1+\sqrt{-1}\tan\theta}{1-\sqrt{-1}\tan\theta}$

$$2\theta\sqrt{-1}=\log(1+\sqrt{-1}\tan\theta)-\log(1-\sqrt{-1}\tan\theta)$$
$$=\sqrt{-1}\tan\theta+\frac{1}{2}\tan^2\theta-\frac{1}{3}\sqrt{-1}\tan^3\theta-\frac{1}{4}\tan^4\theta-\ldots$$
$$-\left(-\sqrt{-1}\tan\theta+\frac{1}{2}\tan^2\theta+\frac{1}{3}\sqrt{-1}\tan^3\theta-\frac{1}{4}\tan^4\theta\ldots\right)$$
$$=2\sqrt{-1}\left(\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta\ldots\right)+\frac{1}{2}\tan^2\theta-\frac{1}{4}\tan^4\theta\ldots$$

Equate impossible parts,

$$\therefore\ \theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\ldots$$

A convergent series if $\tan\theta$ is $=$ or <1, *i.e.* if θ lies between $-\dfrac{\pi}{4}$ and $\dfrac{\pi}{4}$, or between $\dfrac{3\pi}{4}$ and $\dfrac{5\pi}{4}$.

Cor. (i.) if $\tan\theta=x$

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$$

Cor. (ii.) if $\theta=\dfrac{\pi}{4}$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots$$

A series which is very slowly convergent.

$$\text{Also } \frac{\pi}{4}=\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\ldots$$
$$=\frac{2}{1.3}+\frac{2}{5.7}+\frac{2}{9.11}+\ldots$$
$$\text{And } \frac{\pi}{4}=1-\left(\frac{1}{3}-\frac{1}{5}\right)-\left(\frac{1}{7}-\frac{1}{9}\right)-\ldots$$
$$=1-\frac{2}{3.5}-\frac{2}{7.9}-\ldots$$

4. **Prop.** Euler's Series, for determining the value of π.

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\,.\,\frac{1}{3}}=\tan^{-1}1=\frac{\pi}{4}$$

$$\therefore\ \frac{\pi}{4}=\left\{\frac{1}{2}-\frac{1}{3}.\frac{1}{2^3}+\frac{1}{5.2^5}-\ .\ .\ .\right\}+\left\{\frac{1}{3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\ .\ .\ .\right\}$$

5. Machin's Series.

It may be shown that $\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$

$$\therefore\frac{\pi}{4}=4\left\{\frac{1}{5}-\frac{1}{3.5^3}+\frac{1}{5.5^5}-\ .\ .\ .\right\}-\left\{\frac{1}{239}-\frac{1}{3.239^3}+\frac{1}{5.239^5}\ .\ .\ .\right\}$$

Three terms of the first series and one of the second will give the value $\pi=3{\cdot}1416$: eight terms of the first and three of the second will give the value correct to 15 places of decimals.

6. **Prop.** Given two sides and the included angle of a triangle, find the third side and angles, by means of infinite series.

(i.) $c^2=a^2+b^2-2ab\cos C$

$=a^2+b^2-ab\,(e^{C\sqrt{-1}}+e^{-C\sqrt{-1}})$ (lxxiv. 3.)

$=(a-be^{C\sqrt{-1}})\,(a-be^{-C\sqrt{-1}})$

$=a^2\left(1-\frac{b}{a}\,e^{C\sqrt{-1}}\right)\left(1-\frac{b}{a}\,e^{-C\sqrt{-1}}\right)$

$2\log c=2\log a+\log\left(1-\frac{b}{a}\,e^{C\sqrt{-1}}\right)+\log\left(1-\frac{b}{a}\,e^{-C\sqrt{-1}}\right)$

$=2\log a-\frac{b}{a}\,(e^{C\sqrt{-1}}+e^{-C\sqrt{-1}})-\frac{b^2}{2a^2}\,(e^{2C\sqrt{-1}}+e^{-2C\sqrt{-1}})$

$-\frac{b^3}{3a^3}\,(e^{3C\sqrt{-1}}+e^{-3C\sqrt{-1}})-\ .\ .\ .$

$=2\log a-\frac{b}{a}\,2\cos C-\frac{b^2}{2a^2}\,2\cos 2C-\frac{b^3}{3a^3}\,2\cos 3C-\ .\ .\ .\ ,$

$\therefore\ \log c=\log a-\frac{b}{a}\cos C-\frac{b^2}{2a^2}\cos 2C-\frac{b^3}{3a^3}\cos 3C-\ .\ .\ .$

a series which is convergent when $b<a$.

(ii.) Also $\frac{a}{b}=\frac{\sin A}{\sin B}=\frac{\sin (B+C)}{\sin B}=\cos C+\frac{\sin C}{\tan B}$

$$a \tan B=b\cos C\tan B+b \sin C$$

$$\tan B=\frac{b \sin C}{a-b \cos C}$$

$$\frac{e^{2B\sqrt{-1}}-1}{e^{2B\sqrt{-1}}+1}=\frac{b\ (e^{C\sqrt{-1}}-e^{-C\sqrt{-1}})}{2a-b\ (e^{C\sqrt{-1}}+e^{-C\sqrt{-1}})}$$

$$e^{2B\sqrt{-1}}=\frac{a-be^{-C\sqrt{-1}}}{a+be^{C\sqrt{-1}}}=\frac{1-\frac{b}{a}\ e^{-C\sqrt{-1}}}{1+\frac{b}{a}\ e^{C\sqrt{-1}}}$$

$$2B\sqrt{-1}=\log\left(1-\frac{b}{a}\ e^{-C\sqrt{-1}}\right)-\log\left(1+\frac{b}{a}\ e^{C\sqrt{-1}}\right)$$

$$=\frac{b}{a}\ (e^{C\sqrt{-1}}-e^{-C\sqrt{-1}})+\frac{b^2}{2a^2}\ (e^{2C\sqrt{-1}}-e^{-2C\sqrt{-1}})$$

$$+\frac{b^3}{3a^3}\ (e^{3C\sqrt{-1}}-e^{-3C\sqrt{-1}})+\ldots$$

$$=\frac{b}{a}\ 2\sqrt{-1}\ \sin C+\frac{b^2}{2a^2}\ 2\sqrt{-1}\ \sin 2C+\frac{b^3}{3a^3}\ 2\sqrt{-1}\ \sin 3C+\ldots$$

$$\therefore\ B=\frac{b}{a}\ \sin C+\frac{b^2}{2a^2}\sin 2C+\frac{b^3}{3a^3}\sin 3C+\ \ldots$$

a series which is convergent if $b<a$.

7. PROP. Given $\sin x=n\ \sin (x+a)$, expand x in powers of n.

$$e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}=n\ [e^{(x+\alpha)\sqrt{-1}}-e^{-(x+\alpha)\sqrt{-1}}]$$

$$e^{x\sqrt{-1}}\ [1-ne^{\alpha\sqrt{-1}}]=e^{-x\sqrt{-1}}\ [1-ne^{-\alpha\sqrt{-1}}]$$

$$e^{2x\sqrt{-1}}=\frac{1-ne^{-\alpha\sqrt{-1}}}{1-ne^{\alpha\sqrt{-1}}}$$

$$2x\sqrt{-1}=\log\ (1-ne^{-\alpha\sqrt{-1}})-\log\ (1-ne^{\alpha\sqrt{-1}})$$

$$=n\ (e^{\alpha\sqrt{-1}}-e^{-\alpha\sqrt{-1}})+\frac{n^2}{2}\ (e^{2\alpha\sqrt{-1}}-e^{-2\alpha\sqrt{-1}})$$

$$+\frac{n^3}{3}\ (e^{3\alpha\sqrt{-1}}-e^{-3\alpha\sqrt{-1}})+\ \ldots$$

$$=2\sqrt{-1}\ n\sin a+\frac{n^2}{2}\ 2\sqrt{-1}\ \sin 2a+\frac{n^3}{3}\ 2\sqrt{-1}\ \sin 3a+\ \cdots$$

$$\therefore\ x=n\ \sin a+\frac{n^2}{2}\sin 2a+\frac{n^3}{3}\sin 3a+\ \ldots$$

8. **Prop.** Given $\tan x=n\tan a$, expand x in powers of n.

$$\frac{e^{2x\sqrt{-1}}-1}{e^{2x\sqrt{-1}}+1}=n\,\frac{e^{2a\sqrt{-1}}-1}{e^{2a\sqrt{-1}}+1} \qquad \text{(lxxxvii. 4.)}$$

By the theory of proportion,

$$e^{2x\sqrt{-1}}=\frac{(e^{2a\sqrt{-1}}+1)+n\,(e^{2a\sqrt{-1}}-1)}{(e^{2a\sqrt{-1}}+1)-n\,(e^{2a\sqrt{-1}}-1)}$$

$$=\frac{e^{2a\sqrt{-1}}\,(1+n)+(1-n)}{e^{2a\sqrt{-1}}\,(1-n)+(1+n)},\qquad \text{let } m=\frac{1-n}{1+n},$$

$$=\frac{e^{2a\sqrt{-1}}+m}{me^{2a\sqrt{-1}}+1}$$

$$=e^{2a\sqrt{-1}}\left\{\frac{1+me^{-2a\sqrt{-1}}}{1+me^{2a\sqrt{-1}}}\right\}$$

$$2x\sqrt{-1}=2a\sqrt{-1}+\log\,(1+me^{-2a\sqrt{-1}})-\log\,(1+me^{2a\sqrt{-1}})$$

$$=2a\sqrt{-1}-m\,(e^{2a\sqrt{-1}}-e^{-2a\sqrt{-1}})+\frac{m^2}{2}\,(e^{4a\sqrt{-1}}-e^{-4a\sqrt{-1}})-\ldots$$

$$=2a\sqrt{-1}-m\,2\sqrt{-1}\sin 2a+\frac{m^2}{2}\,2\sqrt{-1}\sin 4a-\ldots$$

$$\therefore\ x=a-m\sin 2a+\frac{m^2}{2}\sin 4a-\ldots$$

9. **Prop.** Find the coefficient of x^n in $e^{ax}\cos bx$.

$$e^{ax}\cos bx=\frac{1}{2}\,e^{ax}\,(e^{bx\sqrt{-1}}+e^{-bx\sqrt{-1}})$$

$$=\frac{1}{2}\,\{e^{(a+b\sqrt{-1})x}+e^{(a-b\sqrt{-1})x}\}$$

$$=\frac{1}{2}\left\{1+(a+b\sqrt{-1})x+\frac{(a+b\sqrt{-1})^2x^2}{\underline{|2}}+\ldots+\frac{(a+b\sqrt{-1})^nx^n}{\underline{|n}}+\ldots\right.$$

$$\left.+1+(a-b\sqrt{-1})x+\frac{(a-b\sqrt{-1})^2x^2}{\underline{|2}}+\ldots+\frac{(a-b\sqrt{-1})^nx^n}{\underline{|n}}+\ldots\right\}$$

Coefficient of x^n

$$=\frac{1}{2}\left\{\frac{(a+b\sqrt{-1})^n}{\underline{|n}}+\frac{(a-b\sqrt{-1})^n}{\underline{|n}}\right\}$$

Let $a=r\cos\theta$, and $b=r\sin\theta$, $\therefore\ a^2+b^2=r^2$.

Coefficient of x^n

$$=\frac{r^n}{2\underline{|n}}\left\{\left(\frac{a}{r}+\sqrt{-1}\frac{b}{r}\right)^n+\left(\frac{a}{r}-\frac{b}{r}\sqrt{-1}\right)^n\right\}$$

$$=\frac{r^n}{2\underline{|n}}\left\{(\cos\theta+\sqrt{-1}\sin\theta)^n+(\cos\theta-\sqrt{-1}\sin\theta)^n\right\}$$

$$=\frac{(a^2+b^2)^{\frac{n}{2}}}{2\underline{|n}}\left\{\cos n\theta+\sqrt{-1}\sin n\theta+\cos n\theta-\sqrt{-1}\sin n\theta\right\}$$

$$=\frac{(a^2+b^2)^{\frac{n}{2}}}{2\underline{|n}}\,2\cos n\theta$$

$$=\frac{(a^2+b^2)^{\frac{n}{2}}}{\underline{|n}}\cos n\theta=\frac{(a^2+b^2)^{\frac{n}{2}}}{\underline{|n}}\cos\left\{n\sin^{-1}\frac{b}{\sqrt{a^2+b^2}}\right\}$$

Also coefficient of x^n in $e^{ax}\sin bx$ is $\frac{(a^2+b^2)^{\frac{n}{2}}}{\underline{|n}}\sin n\theta$.

EXAMPLES.

Ex. 1. Reduce $\cos(a+\beta\sqrt{-1})$ to the form $x+y\sqrt{-1}$.

$$\begin{aligned}2\cos(a+\beta\sqrt{-1})&=e^{(\alpha+\beta\sqrt{-1})\sqrt{-1}}+e^{-(\alpha+\beta\sqrt{-1})\sqrt{-1}}\\&=e^{\alpha\sqrt{-1}-\beta}+e^{-\alpha\sqrt{-1}+\beta}\\&=e^{-\beta}\,e^{\alpha\sqrt{-1}}+e^{\beta}\,e^{-\alpha\sqrt{-1}}\\&=e^{-\beta}(\cos a+\sqrt{-1}\sin a)+e^{\beta}(\cos a-\sqrt{-1}\sin a)\\&=\cos a\,(e^{-\beta}+e^{\beta})-\sqrt{-1}\sin a\,(e^{\beta}-e^{-\beta}),\end{aligned}$$

which is the required form.

Ex. 2. If $\log(x+y\sqrt{-1})=a+\beta\sqrt{-1}$, prove that $x^2+y^2=e^{2a}$, and $y=x\tan\beta$.

$$\begin{aligned}x+y\sqrt{-1}&=e^{a+\beta\sqrt{-1}}=e^{a}\,e^{\beta\sqrt{-1}}\\&=e^{a}(\cos\beta+\sqrt{-1}\sin\beta),\end{aligned}$$

equate possible and impossible parts : (lxv. 7.)

$$\left.\begin{aligned}x&=e^{a}\cos\beta\\y&=e^{a}\sin\beta\end{aligned}\right\}$$

$$\therefore\ x^2+y^2=e^{2a}(\sin^2\beta+\cos^2\beta)=e^{2a}.$$

Also $\frac{y}{x}=\tan\beta$.

Ex. 3. Apply the exponential values of $\sin\theta$ and $\cos\theta$ to show that $\cot\frac{\theta}{2}=\frac{\sin\theta}{1-\cos\theta}$.

$$\frac{\sin\theta}{1-\cos\theta}=\frac{\frac{1}{2\sqrt{-1}}\left(e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}\right)}{1-\frac{1}{2}\left(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}\right)} \qquad \text{(lvii. 11.)}$$

$$=\frac{e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{\sqrt{-1}\left(2-e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}\right)}=\frac{\left(e^{\frac{\theta}{2}\sqrt{-1}}-e^{-\frac{\theta}{2}\sqrt{-1}}\right)\left(e^{\frac{\theta}{2}\sqrt{-1}}+e^{-\frac{\theta}{2}\sqrt{-1}}\right)}{-\sqrt{-1}\left(e^{\frac{\theta}{2}\sqrt{-1}}-e^{-\frac{\theta}{2}\sqrt{-1}}\right)^2}$$

$$=\frac{e^{\frac{\theta}{2}\sqrt{-1}}+e^{-\frac{\theta}{2}\sqrt{-1}}}{-\sqrt{-1}\left(e^{\frac{\theta}{2}\sqrt{-1}}-e^{-\frac{\theta}{2}\sqrt{-1}}\right)}=\frac{2\cos\frac{\theta}{2}}{-\sqrt{-1}\left\{2\sqrt{-1}\sin\frac{\theta}{2}\right\}}=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=\cot\frac{\theta}{2}.$$

Ex. 4. Prove that $(3^{\frac{1}{2}}-3^{-\frac{1}{2}})-\frac{1}{3}(3^{\frac{3}{2}}-3^{-\frac{3}{2}})+\frac{1}{5}(3^{\frac{5}{2}}-3^{-\frac{5}{2}})-\ldots=\frac{\pi}{6}$.

By **Gregory's** series,

$$\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\ldots$$

$$\therefore\ \frac{\pi}{3}=\tan\frac{\pi}{3}-\frac{1}{3}\tan^3\frac{\pi}{3}+\frac{1}{5}\tan^5\frac{\pi}{3}-\ldots \qquad \text{(lxii. 3.)}$$

$$=3^{\frac{1}{2}}-\frac{1}{3}\cdot 3^{\frac{3}{2}}+\frac{1}{5}\cdot 3^{\frac{5}{2}}-\ldots \qquad \text{(i.)}$$

$$\text{Also } \frac{\pi}{6}=\tan\frac{\pi}{6}-\frac{1}{3}\tan^3\frac{\pi}{6}+\frac{1}{5}\tan^5\frac{\pi}{6}-\ldots$$

$$=3^{-\frac{1}{2}}-\frac{1}{3}\,3^{-\frac{3}{2}}+\frac{1}{5}\,3^{-\frac{5}{2}}-\ldots \qquad \text{(ii.)}$$

Subtract (ii.) from (i.),

$$\frac{\pi}{6}=(3^{\frac{1}{2}}-3^{-\frac{1}{2}})-\frac{1}{3}(3^{\frac{3}{2}}-3^{-\frac{3}{2}})+\frac{1}{5}(3^{\frac{5}{2}}-3^{-\frac{5}{2}})\ldots$$

Ex. 5. If $\sin(\alpha+\beta\sqrt{-1})=x+y\sqrt{-1}$, prove $x^2\operatorname{cosec}^2\alpha-y^2\sec^2\alpha=1$.

$$\sin\alpha\cos\beta\sqrt{-1}+\cos\alpha\sin\beta\sqrt{-1}=x+y\sqrt{-1}$$

$$\frac{1}{2}\sin\alpha\,(e^{-\beta}+e^{\beta})+\frac{1}{2\sqrt{-1}}\cos\alpha\,(e^{-\beta}-e^{\beta})=x+y\sqrt{-1}$$

$$\sqrt{-1}\sin\alpha\,(e^{-\beta}+e^{\beta})+\cos\alpha\,(e^{-\beta}-e^{\beta})=2\sqrt{-1}\,x-2y$$

$$2x=\sin\alpha\,(e^{-\beta}+e^{\beta})$$

(lxxv. 4.)
$$2y=-\cos\alpha\,(e^{-\beta}-e^{\beta})$$

$$2x\operatorname{cosec}\alpha=e^{-\beta}+e^{\beta}$$

$$2y\sec\alpha=e^{\beta}-e^{-\beta}$$

$$4x^2\operatorname{cosec}^2\alpha-4y^2\sec^2\alpha=2+2=4$$

$$\therefore\ x^2\operatorname{cosec}^2\alpha-y^2\sec^2\alpha=1.$$

Ex. 6. Prove $2\cos(n\cos^{-1}x)=(x+\sqrt{-1}\sqrt{1-x^2})^n+(x-\sqrt{-1}\sqrt{1-x^2})^n$.

Let $n\cos^{-1}x=\theta$ (lxxxv. 6.)

$$\therefore\ x=\cos\frac{\theta}{n},\ \text{and}\ \sin\frac{\theta}{n}=\sqrt{1-x^2}.$$

$$\text{and } 2\cos\theta=(\cos\theta+\sqrt{-1}\sin\theta)+(\cos\theta-\sqrt{-1}\sin\theta)$$

$$2\cos(n\cos^{-1}x)=\left(\cos\frac{\theta}{n}+\sqrt{-1}\sin\frac{\theta}{n}\right)^n+\left(\cos\frac{\theta}{n}-\sqrt{-1}\sin\frac{\theta}{n}\right)^n$$

$$\therefore\ 2\cos(n\cos^{-1}x)=(x+\sqrt{-1}\sqrt{1-x^2})^n+(x-\sqrt{-1}\sqrt{1-x^2})^n.$$

Ex. 7. Prove that $(\sqrt{-1})^{\sqrt{-1}}=e^{-\frac{\pi}{2}}$.

$$\sqrt{-1}=\left(\cos\frac{\pi}{2}+\sqrt{-1}\sin\frac{\pi}{2}\right)$$ (lxxxix. 3.)

$$=e^{\frac{\pi}{2}\sqrt{-1}}$$

$$\therefore\ (\sqrt{-1})^{\sqrt{-1}}=\left(e^{\frac{\pi}{2}\sqrt{-1}}\right)^{\sqrt{-1}}$$

$$=e^{-\frac{\pi}{2}}.$$

Ex. 8. Find the co-efficients of x^n in the expansion of $e^x \cos x$.

$$e^x \cos x = \frac{1}{2} e^x \left(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right) \qquad \text{(cxxiii. 8.)}$$

$$= \frac{1}{2} \left(e^{x(1+\sqrt{-1})} + e^{x(1-\sqrt{-1})}\right).$$

$$\text{Coefficient of } x^n = \frac{1}{2} \left\{ \frac{(1+\sqrt{-1})^n}{\underline{|n}} + \frac{(1-\sqrt{-1})^n}{\underline{|n}} \right\}$$

$$= \frac{1}{2\underline{|n}} \left\{ \left(\cos\frac{\pi}{4} + \sqrt{-1}\sin\frac{\pi}{4}\right)^n + \left(\cos\frac{\pi}{4} - \sqrt{-1}\sin\frac{\pi}{4}\right)^n \right\} 2^{\frac{n}{2}}$$

$$= \frac{2^{\frac{n}{2}}}{2\underline{|n}} \left\{ \cos\frac{n\pi}{4} + \sqrt{-1}\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - \sqrt{-1}\sin\frac{n\pi}{4} \right\}$$

$$= \frac{2^{\frac{n}{2}}}{\underline{|n}} \cos\frac{n\pi}{4}.$$

Ex. 9. Prove that $e^\theta(\cos\theta + \sqrt{-1}\sin\theta) = e^{\theta\sqrt{2}\left(\cos\frac{\pi}{4} + \sqrt{-1}\sin\frac{\pi}{4}\right)}$.

Hence prove that the coefficient of θ^{2n} in the expansions of $e^\theta\cos\theta$ and $e^\theta\sin\theta$ will be $\frac{2^n}{\underline{|2n}}\cos\frac{n\pi}{2}$ and $\frac{2^n}{\underline{|2n}}\sin\frac{n\pi}{2}$ respectively.

$$\cos\theta + \sqrt{-1}\sin\theta = e^{\theta\sqrt{-1}}$$

$$\therefore\ e^\theta(\cos\theta + \sqrt{-1}\sin\theta) = e^{\theta(1+\sqrt{-1})} = e^{\theta\sqrt{2}\left(\cos\frac{\pi}{4} + \sqrt{-1}\sin\frac{\pi}{4}\right)} \qquad \text{(lxiv. 8.)}$$

$$\therefore\ e^\theta\cos\theta + \sqrt{-1}\,e^\theta\sin\theta = 1 + \theta\sqrt{2}\left(\cos\frac{\pi}{4} + \sqrt{-1}\sin\frac{\pi}{4}\right) + \frac{1}{2}\left(\theta\sqrt{2}\left(\cos\frac{\pi}{4} + \sqrt{-1}\sin\frac{\pi}{4}\right)\right)^2 + \ldots$$

$$\text{coefficient of } \theta^{2n} \text{ is } \frac{1}{\underline{|2n}} \left\{ \sqrt{2}\left(\cos\frac{\pi}{4} + \sqrt{-1}\sin\frac{\pi}{4}\right) \right\}^{2n}$$

$$= \frac{2^n}{\underline{|2n}} \left\{ \cos\frac{n\pi}{2} + \sqrt{-1}\sin\frac{n\pi}{2} \right\}.$$

$\therefore$ coefficient of θ^{2n} in $e^\theta\cos\theta$ is $\frac{2^n}{\underline{|2n}}\cos\frac{n\pi}{2}$,

and coefficient of θ^{2n} in $e^\theta\sin\theta$ is $\frac{2^n}{\underline{|2n}}\sin\frac{n\pi}{2}$.

Ex. 10. Show that every number, positive or negative, has an infinite number of imaginary logarithms.

$$\cos x + \sqrt{-1}\,\sin x = e^{x\sqrt{-1}}.$$

For x write $2r\pi + x$, then

$$\cos(2r\pi + x) + \sqrt{-1}\,\sin(2r\pi + x) = e^{(2r\pi + x)\sqrt{-1}}.$$

$x = 0$, gives $1 = e^{2r\pi\sqrt{-1}}$,

$x = \pi$, gives $-1 = e^{(2r+1)\pi\sqrt{-1}}$ (cxxiv. 6.)

$$\therefore \log_e 1 = 2r\pi\sqrt{-1}$$

$$\log_e(-1) = (2r+1)\pi\sqrt{-1},$$

if $r = 0$, then $\log_e 1 = 0$, which is the common Arithmetical Logarithm of unity, but as r may be any whole number, $+1$ has an infinite number of Napierian logarithms, all of which except 0 are imaginary.

Hence also it follows that every positive number has an infinite number of logarithms to the same base;

For suppose $\log_e y = x$, then $y = e^x \times 1$

$$= e^x \times e^{2r\pi\sqrt{-1}}$$

$$= e^{(x + 2r\pi\sqrt{-1})}$$

$$\therefore \log_e y = x + 2r\pi\sqrt{-1},$$

and as we can transfer the base to 10, or to any base, we conclude that every number has an infinite number of logarithms to that base.

Again, in

$$\log_e(-1) = (2r+1)\pi\sqrt{-1},$$

since r is to be an integer, $\therefore \log_e(-1)$ can never be a possible quantity, and therefore every negative number has an infinite number of logarithms to any base, all of which are affected with $\sqrt{-1}$.

XXVI.

Resolution of Expressions into Factors.

1. PROP. Resolve $x^n - 1$ into factors.

$$\text{If } x^n - 1 = 0, \qquad x^n = 1 = \cos 2r\pi \pm \sqrt{-1}\,\sin 2r\pi,$$

$$\therefore x = (\cos 2r\pi \pm \sqrt{-1}\,\sin 2r\pi)^{\frac{1}{n}} \quad = \cos\frac{2r\pi}{n} \pm \sqrt{-1}\,\sin\frac{2r\pi}{n}.$$

(i.) n even, put $r = 0, 1, 2, \ldots \frac{n}{2}$ in the factors, which are

$$\left(x - \cos\frac{2r\pi}{n} - \sqrt{-1}\,\sin\frac{2r\pi}{n}\right)\left(x - \cos\frac{2r\pi}{n} + \sqrt{-1}\,\sin\frac{2r\pi}{n}\right)$$

$$= \left(x - \cos\frac{2r\pi}{n}\right)^2 + \sin^2\frac{2r\pi}{n} \qquad \text{liv. 8.)}$$

$$= x^2 - 2x\cos\frac{2r\pi}{n} + 1, \text{ and we get}$$

$$x^n - 1 = (x-1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\ldots$$
$$\left(x^2 - 2x\cos\frac{(n-2)\pi}{n} + 1\right)\left(x+1\right).$$

(ii.) n odd, put $r = 0, 1, 2, \ldots \frac{n-1}{2}$, and we get

$$x^n - 1 = (x-1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\ldots.$$
$$\left(x^2 - 2x\cos\frac{(n-1)\pi}{n} + 1\right).$$

2. PROP. Resolve $x^n + 1$ into factors.

$$\text{If } x^n + 1 = 0,\ x^n = -1 = \cos(2r\pi + \pi) \pm \sqrt{-1}\,\sin(2r\pi + \pi)$$

$$\therefore x = (\cos(2r+1)\pi \pm \sqrt{-1}\,\sin(2r+1)\pi)^{\frac{1}{n}}$$

$$= \cos\frac{(2r+1)\pi}{n} \pm \sqrt{-1}\,\sin\frac{(2r+1)\pi}{n}$$

(i.) n even, put $r = 0, 1, 2, \ldots \frac{n}{2} - 1$ in the factors, which are

$$\left(x - \cos\frac{(2r+1)\pi}{n} - \sqrt{-1}\,\sin\frac{(2r+1)\pi}{n}\right)\left(x - \cos\frac{(2r+1)\pi}{n} + \sqrt{-1}\,\sin\frac{(2r+1)\pi}{n}\right)$$

$$= x^2 - 2x\cos\frac{(2r+1)\pi}{n} + 1, \text{ and we get}$$

$$x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right)\ldots.$$
$$\left(x^2 - 2x\cos\frac{(n-1)\pi}{n} + 1\right).$$

(ii.) n odd, put $r=0, 1, 2, \ldots \frac{n-1}{2}$, and we get

$$x^n+1=(x+1)\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{3\pi}{n}+1\right)\ldots\ldots$$
$$\left(x^2-2x\cos\frac{(n-2)\pi}{n}+1\right).$$

3. PROP. Resolve $x^{2n}-2x^n\cos\theta+1$ into factors.

Let $x^{2n}-2x^n\cos\theta+1=0$, $\quad x^{2n}-2x^n\cos\theta+\cos^2\theta=\cos^2\theta-1=-\sin^2\theta$

$$x^n-\cos\theta=\pm\sqrt{-1}\sin\theta, \qquad x^n=\cos\theta\pm\sqrt{-1}\sin\theta$$

$$\therefore x=(\cos\theta\pm\sqrt{-1}\sin\theta)^{\frac{1}{n}}=\cos\frac{2r\pi+\theta}{n}\pm\sqrt{-1}\sin\frac{2r\pi+\theta}{n},$$

put $r=0, 1, 2, \ldots n-1$, we get $2n$ different values, and the factors are

$$\left(x-\cos\frac{2r\pi+\theta}{n}-\sqrt{-1}\sin\frac{2r\pi+\theta}{n}\right)\left(x-\cos\frac{2r\pi+\theta}{n}+\sqrt{-1}\sin\frac{2r\pi+\theta}{n}\right)$$
$$=\left(x-\cos\frac{2r\pi+\theta}{n}\right)^2+\sin^2\frac{2r\pi+\theta}{n}=x^2-2x\cos\frac{2r\pi+\theta}{n}+1,$$

$$\therefore x^{2n}-2x^n\cos\theta+1=\left(x^2-2x\cos\frac{\theta}{n}+1\right)\left(x^2-2x\cos\frac{2\pi+\theta}{n}+1\right)\ldots$$
$$\left(x^2-2x\cos\frac{2(n-1)\pi+\theta}{n}+1\right).$$

4. Since $\frac{x^n-1}{x-1}=x^{n-1}+x^{n-2}+x^{n-3}+\ldots+x+1=n$, when $x=1$.

By (ii.) in PROP. 1.

$$\therefore n=\left(2-2\cos\frac{2\pi}{n}\right)\left(2-2\cos\frac{4\pi}{n}\right)\left(2-2\cos\frac{6\pi}{n}\right)\ldots\left(2-2\cos\frac{(n-1)\pi}{n}\right)$$
$$=2^2\sin^2\frac{\pi}{n}.\,2^2\sin^2\frac{2\pi}{n}\ldots 2^2\sin^2\frac{(n-1)\pi}{2n}$$
$$=2^{n-1}\sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{n}\ldots\sin^2\frac{(n-1)\pi}{n}$$
$$\therefore \sqrt{n}=2^{\frac{n-1}{2}}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}.$$

taking the + sign of the radical, since every sine is +, each angle being $<\frac{\pi}{2}$.

5. PROP. Resolve $\sin\theta$ into the product of quadratic factors in terms of θ.

If $\sin\theta=0$, $\theta=n\pi$, where n is zero or any positive or negative integer,

$$\therefore\ \theta=0,\ \pm\pi,\ \pm2\pi,\ \pm3\pi,\ \ldots \text{ are the solutions}$$

$$\sin\theta=a\theta\,(\theta+\pi)\,(\theta-\pi)\,(\theta+2\pi)\,(\theta-2\pi)\ \ldots$$

$$=a\theta\,(\theta^2-\pi^2)\,(\theta^2-2^2\pi^2)\,(\theta^2-3^2\pi^2)\ \ldots \qquad \text{(i.)}$$

Where a = some constant quantity.

Now (i.) may be written, (lxxviii. 5.)

$$\frac{\sin\theta}{\theta}=a\,(\theta^2-\pi^2)\,(\theta^2-2^2\pi^2)\,(\theta^2-3^2\pi^2)\ \ldots,$$

which is true when $\theta=0$, and then $\frac{\sin\theta}{\theta}=1$,

$$\therefore\ 1=a\,(-\pi^2)\,(-2^2\pi^2)\,(-3^2\pi^2)\ \ldots$$

$$1=\pm a\pi^2\,.\,2^2\pi^2\,.\,3^2\pi^2\ \ldots \qquad \text{(ii.)}$$

$$\text{from (i.),}\quad \sin\theta=\pm a\pi^2\,.\,2^2\pi^2\,.\,3^2\pi^2\ldots\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots$$

$$\text{from (ii.),}\quad \sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ \ldots$$

when $\theta>0$ and $<\pi$, $\sin\theta$ is +, and each factor on the right hand is + when $\theta>\pi$ and $<2\pi$, $\sin\theta$ is −, and one factor only on the right hand is −, and so on.

Hence the + sign must be taken, and there is no ambiguity.

COR. **Wallis's Expression for π.**

Let $\theta=\frac{\pi}{2}$, then $\sin\frac{\pi}{2}=1$, and $\frac{\theta}{\pi}=\frac{1}{2}$,

$$1=\frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{2^2.2^2}\right)\left(1-\frac{1}{3^2.2^2}\right)\ \ldots$$

$$=\frac{\pi}{2}\ \frac{1.3}{2^2}\cdot\frac{3.5}{4^2}\cdot\frac{5.7}{6^2}\ \ldots$$

$$=\frac{\pi}{2}\cdot\frac{3^2.5^2.7^2.9^2\ldots}{2^2.4^2.6^2.8^2\ldots}$$

$$\therefore\ \frac{\pi}{2}=\frac{2^2.4^2.6^2.8^2\ldots}{3^2.5^2.7^2.9^2\ldots}$$

6. PROP. Resolve $\cos\theta$ into the product of quadratic factors in terms of θ.

If $\cos\theta=0$, $\theta=2n\pi\pm\frac{\pi}{2}$, where n is zero or any positive or negative integer,

$$\therefore\ \theta=\pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\ldots \text{ are the solutions,}$$

$$\therefore\ \cos\theta=a\left(\theta+\frac{\pi}{2}\right)\left(\theta-\frac{\pi}{2}\right)\left(\theta+\frac{3\pi}{2}\right)\left(\theta-\frac{3\pi}{2}\right)\ldots$$

$$=a\left(\theta^2-\frac{\pi^2}{2^2}\right)\left(\theta^2-\frac{3^2\pi^2}{2^2}\right)\ldots\quad .\quad .\quad .\quad .\qquad \text{(i.)}$$

where a = some constant quantity.

If $\theta=0$, $\cos\theta=1$

$$\therefore\ 1=a\left(-\frac{\pi^2}{2^2}\right)\left(-\frac{3^2\pi^2}{2^2}\right)\left(-\frac{5^2\pi^2}{2^2}\right)\ldots$$

$$=\pm a\,.\,\frac{\pi^2}{2^2}\cdot\frac{3^2\pi^2}{2^3}\cdot\frac{5^2\pi^2}{2^2}\ldots\quad .\quad .\quad .\quad .\qquad \text{(ii.)}$$

from (i.) $\cos\theta=\pm a\frac{\pi^2}{2^2}\cdot\frac{3^2\pi^2}{2^2}\cdot\frac{5^2\pi^2}{2^2}\cdots\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\cdots$

from (ii.) $\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots$

EXAMPLES.

Ex. 1. Resolve $e^\theta\pm e_{-\theta}$ into quadratic factors.

$$e^{a\sqrt{-1}}+e^{-a\sqrt{-1}}=2\cos a=2\left(1-\frac{4a^2}{\pi^2}\right)\left(1-\frac{4a^2}{3^2\pi^2}\right)\ldots$$

$$e^{a\sqrt{-1}}-e^{-a\sqrt{-1}}=2\sqrt{-1}\ \sin a=2a\sqrt{-1}\left(1-\frac{a^2}{\pi^2}\right)\left(1-\frac{a^2}{2^2\pi^2}\right)\ldots$$

Let $a\sqrt{-1}=\theta$,

$$\therefore\ e^\theta+e^{-\theta}=2\left(1+\frac{4\theta^2}{\pi^2}\right)\left(1+\frac{4\theta^2}{3^2\pi^2}\right)\ldots$$

$$e^\theta-e^{-\theta}=2\theta\left(1+\frac{\theta^2}{\pi^2}\right)\left(1+\frac{\theta^2}{2^2\pi^2}\right)\ldots$$

Ex. 2. Having given the series for $\sin\theta$ and $\cos\theta$ in powers of θ, show that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6},\text{ and }\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}.$$

$$\sin\theta=\theta-\frac{\theta^3}{\underline{|3}}+\frac{\theta^5}{\underline{|5}}-\ldots=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$$

$$\therefore 1-\frac{\theta^2}{\underline{|3}}+\frac{\theta^4}{\underline{|5}}-\ldots=1-\frac{\theta^2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)\qquad\text{(lxxxv. 7.)}$$

$$+\frac{\theta^4}{\pi^4}\left(\frac{1}{1^2.2^2}+\frac{1}{1^2.3^2}+\ldots\right)-\ldots$$

Equate coefficients of θ^2,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}.$$

$$\cos\theta=1-\frac{\theta^2}{\underline{|2}}+\frac{\theta^4}{\underline{|4}}-\ldots=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots$$

$$=1-\frac{4\theta^2}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)\ldots$$

Equate coefficients of θ^2,

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}.$$

XXVII.

Summation of Trigonometrical Series.

1. **Prop.** $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots$ to n terms.

$$\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{\beta}{2}\right)=2\sin\alpha\sin\frac{\beta}{2}$$

$$\cos\left(\alpha+\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{3\beta}{2}\right)=2\sin(\alpha+\beta)\sin\frac{\beta}{2}$$

$$\text{etc.}=\text{etc.}\qquad\text{(xcviii. 8.)}$$

$$\therefore S_n=\frac{\sin\left(\alpha+\frac{(n-1)\beta}{2}\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

N.B.—S_∞ cannot be found, since the terms of the series neither increase nor decrease *ad inf.*; and since they are all finite, and are + or − according to the magnitude of the angle, the series will alternately converge and diverge.

2. Sin $a - \sin(a+\beta) + \sin(a+2\beta) - \ldots$ to n terms.

i.e., $\sin a + \sin(\pi + a + \beta) + \sin(2\pi + a + 2\beta) + \ldots$

write $\pi + \beta$ for β in (1),

$$S_n = \frac{\sin\left(a + \frac{(n-1)(\pi+\beta)}{2}\right)\sin\frac{n(\pi+\beta)}{2}}{\sin\frac{\pi+\beta}{2}}.$$

3. Sin $a + \sin 2a + \sin 3a + \ldots$ to n terms.

write $\beta = a$ in (1),

$$S_n = \frac{\sin\frac{(n+1)a}{2}\sin\frac{na}{2}}{\sin\frac{a}{2}}. \qquad \text{(lxxv. 5.)}$$

4. Sin $a + \sin 3a + \sin 5a + \ldots$ to n terms.

write $\beta = 2a$ in (1),

$$S_n = \frac{\sin^2 na}{\sin a}. \qquad \text{(xcvi. 5.)}$$

5. **Prop.** Cos $a + \cos(a+\beta) + \cos(a+2\beta) + \ldots$ to n terms.

$$\sin\left(a + \frac{\beta}{2}\right) - \sin\left(a - \frac{\beta}{2}\right) = 2\cos a \sin\frac{\beta}{2} \qquad \text{(xciii. 7.)}$$

$$\sin\left(a + \frac{3\beta}{2}\right) - \sin\left(a + \frac{\beta}{2}\right) = 2\cos(a+\beta)\sin\frac{\beta}{2}$$

etc. = etc.

$$\therefore S_n = \frac{\cos\left(a + \frac{(n-1)\beta}{2}\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

6. Cos $a - \cos(a+\beta) + \cos(a+2\beta) - \ldots$ to n terms.

i.e., $\cos a + \cos(\pi + a + \beta) + \cos(2\pi + a + 2\beta) + \ldots$

write $\pi + \beta$ for β in (5),

$$S_n = \frac{\cos\left(a + \frac{(n-1)(\pi+\beta)}{2}\right)\sin\frac{n(\pi+\beta)}{2}}{\sin\frac{\pi+\beta}{2}}.$$

7. Cos $\alpha+\cos 2\alpha+\cos 3\alpha+\ldots$ to n terms.

write $\beta=\alpha$ in (5),

$$S_n=\frac{\cos\frac{(n+1)\alpha}{2}\ \sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}. \qquad \text{(lxxxvi. 8.)}$$

8. Cos $\alpha+\cos 3\alpha+\cos 5\alpha+\ldots$ to n terms.

write $\beta=2\alpha$ in (5),

$$S_n=\frac{\sin 2n\alpha}{2\sin\alpha}. \qquad \text{(lxxx. 5.)}$$

9. Sin$^2\alpha+\sin^2(\alpha+\beta)+\sin^2(\alpha+2\beta)+\ldots$ to n terms.

$$2\,S=1-\cos 2\alpha+1-\cos 2(\alpha+\beta)+1-\cos 2(\alpha+2\beta)+\ldots$$

$$=n-(\cos 2\alpha+\cos 2(\alpha+\beta)+\cos 2(\alpha+2\beta)+\ldots)$$

$$S_n=\frac{n\sin\beta-\cos(2\alpha+(n-1)\beta)\sin n\beta}{2\sin\beta}.$$

10. Sin$^2\alpha+\sin^2 2\alpha+\sin^2 3\alpha+\ldots$ to n terms.

write $\beta=\alpha$ in (9),

$$S_n=\frac{n\sin\alpha-\sin n\alpha\cos(n+1)\alpha}{2\sin\alpha}. \qquad \text{(xciii. 7.)}$$

11. Sin$^3\alpha+\sin^3(\alpha+\beta)+\sin^3(\alpha+2\beta)+\ldots$ to n terms.

$$S=\frac{1}{4}(3\sin\alpha-\sin 3\alpha)+\frac{1}{4}(3\sin(\alpha+\beta)-\sin 3(\alpha+\beta))+\ldots$$

$$4\,S=3\ (\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots)$$

$$-(\sin 3\alpha+\sin 3(\alpha+\beta)+\sin 3(\alpha+2\beta)+\ldots)$$

$$S_n=\frac{3}{4}\,\frac{\sin\frac{n\beta}{2}\sin\left(\alpha+\frac{(n-1)\beta}{2}\right)}{\sin\frac{\beta}{2}}-\sin\frac{\frac{3n\beta}{2}\sin 3\left(\alpha+\frac{(n-1)\beta}{2}\right)}{4\sin\frac{3\beta}{2}}.$$

12. $\text{Sin}^3 \alpha+\sin^3 2\alpha+\sin^3 3\alpha+\ldots$ to n terms.

write $\beta=\alpha$ in (11),

$$S_n=\frac{3\sin\frac{n\alpha}{2}\sin\frac{(n+1)\alpha}{2}}{4\sin\frac{\alpha}{2}}-\frac{\sin\frac{3n\alpha}{2}\sin\frac{3(n+1)\alpha}{2}}{4\sin\frac{3\alpha}{2}}.$$

13. $\text{Sin}\,\alpha\sin 2\alpha+\sin 2\alpha\sin 3\alpha+\sin 3\alpha\sin 4\alpha+\ldots$ to n terms.

$$\cos\alpha-\cos 3\alpha=2\sin 2\alpha\sin\alpha$$
$$\cos\alpha-\cos 5\alpha=2\sin 3\alpha\sin 2\alpha$$
$$\text{etc.}=\text{etc.}$$
$$S_n=\frac{n\sin\alpha\cos\alpha-\sin n\alpha\cos(n+2)\alpha}{2\sin\alpha}.$$

14. $\text{Cosec}\,\alpha+\text{cosec}\,2\alpha+\text{cosec}\,2^2\alpha+\ldots$ to n terms.

$$\text{cosec}\,\alpha=\cot\frac{\alpha}{2}-\cot\alpha$$
$$\text{cosec}\,2\alpha=\cot\alpha-\cot 2\alpha$$
$$\text{etc.}=\text{etc.}$$
$$S_n=\cot\frac{\alpha}{2}-\cot 2^{n-1}\alpha.$$

15. $\text{Tan}\,\alpha+2\tan 2\alpha+2^2\tan 2^2\alpha+\ldots$ to n terms.

$$\tan\alpha=\cot\alpha-2\cot 2\alpha$$
$$2\tan 2\alpha=2\cot 2\alpha-2^2\cot 2^2\alpha$$
$$\text{etc.}=\text{etc.}\qquad\text{(lxvi. 6.)}$$
$$S_n=\cot\alpha-2^n\cot 2^n\alpha.$$

16. $\text{Tan}\,\alpha+\frac{1}{2}\tan\frac{\alpha}{2}+\frac{1}{2^2}\tan\frac{\alpha}{2^2}+\ldots$ to n terms.

$$\tan\alpha=\cot\alpha-2\cot 2\alpha$$
$$\frac{1}{2}\tan\frac{\alpha}{2}=\frac{1}{2}\cot\frac{\alpha}{2}-\cot\alpha$$
$$\text{etc.}=\text{etc.}$$
$$S_n=\frac{1}{2^{n-1}}\cot\frac{\alpha}{2^{n-1}}-2\cot 2\alpha.$$

17. $(\tan a+\cot a)+(\tan 2a+\cot 2a)+(\tan 2^2a+\cot 2^2a+)\ldots$ to n terms.

$$\tan a+\cot a=2\cot a-2\cot 2a$$
$$\tan 2a+\cot 2a=2\cot 2a-2\cot 2^2a$$
$$\text{etc.}=\text{etc.}$$

$S_n=2\cot a-2\cot 2^na.$

18. $\text{Sec}\, a \sec 2a+\sec 2a \sec 3a+\ldots$ to n terms.

$$\sec a\sec 2a=\operatorname{cosec} a\,(\tan 2a-\tan a)$$
$$\sec 2a\sec 3a=\operatorname{cosec} a\,(\tan 3a-\tan 2a)$$
$$\text{etc.}=\text{etc.}$$

$S_n=\operatorname{cosec} a\,(\tan(n+1)a-\tan a).$

19. $\text{Cosec}\, a \operatorname{cosec} 2a+\operatorname{cosec} 2a\operatorname{cosec} 3a+\ldots$ to n terms.

$$\operatorname{cosec} a\operatorname{cosec} 2a=\operatorname{cosec} a\,(\cot a-\cot 2a)$$
$$\operatorname{cosec} 2a\operatorname{cosec} 3a=\operatorname{cosec} a\,(\cot 2a-\cot 3a)$$
$$\text{etc.}=\text{etc.}$$

$S_n=\operatorname{cosec} a\,(\cot a-\cot(n+1)a).$

20. $\dfrac{\sin 2\theta}{\cos\theta\cos 3\theta}+\dfrac{\sin 4\theta}{\cos 3\theta\cos 5\theta}+\ldots$ to n terms.

$$2S\sin\theta=\frac{\cos\theta-\cos 3\theta}{\cos\theta\cos 3\theta}+\frac{\cos 3\theta-\cos 5\theta}{\cos 3\theta\cos 5\theta}+\ldots$$

$=(\sec 3\theta-\sec\theta)+(\sec 5\theta-\sec 3\theta)+\ldots+(\sec(2n+1)\theta-\sec(2n-1)\theta)$

$=\sec(2n+1)\theta-\sec\theta$

$$S_n=\frac{\sec(2n+1)\theta-\sec\theta}{2\sin\theta}.$$

21. $\text{Tan}^{-1}\dfrac{1}{1+1+1^2}+\tan^{-1}\dfrac{1}{1+2+2^2}+\ldots$ to n terms.

$$\tan^{-1}\frac{1}{1+1+1^2}=\tan^{-1}\frac{1}{1}-\tan^{-1}\frac{1}{2}$$

$$\tan^{-1}\frac{1}{1+2+2^2}=\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{3}$$

$$\text{etc.}=\text{etc.}$$

$$\therefore\ S_n=\tan^{-1}1-\tan^{-1}\frac{1}{n+1}$$

$$=\frac{\pi}{4}-\tan^{-1}\frac{1}{n+1}.$$

Summation by Exponential Theorem.

22. **Prop.** Let $C=x\cos(\alpha+\beta)+\dfrac{x^2}{\underline{|2}}\cos(\alpha+2\beta)$

$$+\frac{x^3}{\underline{|3}}\cos(\alpha+3\beta)+\ldots\text{ to }\infty.$$

$$S=x\sin(\alpha+\beta)+\frac{x^2}{\underline{|2}}\sin(\alpha+2\beta)$$

$$+\frac{x^3}{\underline{|3}}\sin(\alpha+3\beta)+\ldots\text{ to }\infty.$$

$$C+\sqrt{-1}\,S=xe^{(\alpha+\beta)\sqrt{-1}}+\frac{x^2}{\underline{|2}}e^{(\alpha+2\beta)\sqrt{-1}}+\frac{x^3}{\underline{|3}}e^{(\alpha+3\beta)\sqrt{-1}}+\ldots$$

$$=e^{\alpha\sqrt{-1}}\left\{xe^{\beta\sqrt{-1}}+\frac{x^2}{\underline{|2}}e^{2\beta\sqrt{-1}}+\frac{x^3}{\underline{|3}}e^{3\beta\sqrt{-1}}+\ldots\right\}$$

$$=e^{\alpha\sqrt{-1}}\left\{e^{xe^{\beta\sqrt{-1}}}-1\right\}$$

$$=e^{\alpha\sqrt{-1}}\left\{e^{x(\cos\beta+\sqrt{-1}\sin\beta)}-1\right\}$$

$$=e^{x\cos\beta}\,e^{(\alpha+x\sin\beta)\sqrt{-1}}-e^{\alpha\sqrt{-1}}$$

$$=e^{x\cos\beta}\{\cos(\alpha+x\sin\beta)+\sqrt{-1}\sin(\alpha+x\sin\beta)\}$$

$$-(\cos\alpha+\sqrt{-1}\sin\alpha)$$

Equate possible and impossible parts,

$$C=e^{x\cos\beta}\cos(\alpha+x\sin\beta)-\cos\alpha.$$

$$S=e^{x\cos\beta}\sin(\alpha+x\sin\beta)-\sin\alpha.$$

23. $S=\sin\theta+\frac{\sin\theta}{1}\sin 2\theta+\frac{\sin^2\theta}{1.2}\sin 3\theta+\ldots$ to ∞.

$$2\sqrt{-1}\,S=\left(x-\frac{1}{x}\right)+\sin\theta\left(x^2-\frac{1}{x^2}\right)+\frac{\sin^2\theta}{1.2}\left(x^3-\frac{1}{x^3}\right)+\ldots$$

$$=x\left(1+x\sin\theta+\frac{x^2\sin 2\theta}{1.2}+\ldots\right)-\frac{1}{x}\left(1+\frac{\sin\theta}{x}+\frac{\sin^2\theta}{1.2x^2}+\ldots\right)$$

$$=x\,e^{x\sin\theta}-\frac{1}{x}\;e^{\frac{\sin\theta}{x}}$$

$$=x\,e^{\sin\theta(\cos\theta+\sqrt{-1}\sin\theta)}-\frac{1}{x}\,e^{\sin\theta\;\cos\theta-\sqrt{-1}\sin\theta}$$

$$=e^{\sin\theta\cos\theta}\left[xe^{\sqrt{-1}\sin^2\theta}-\frac{1}{x}e^{-\sqrt{-1}\sin^2\theta}\right]$$

$$=e^{\sin\theta\cos\theta}[\,e^{\theta\sqrt{-1}}.\,e^{\sqrt{-1}\sin^2\theta}-e^{-\theta\sqrt{-1}}.\,e^{-\sqrt{-1}\sin^2\theta}\,]$$

$$=e^{\sin\theta\cos\theta}[\,e^{\sqrt{-1}(\theta+\sin^2\theta)}-e^{-\sqrt{-1}(\theta+\sin^2\theta)}\,]$$

$$=e^{\sin\theta\cos\theta}.\,2\sqrt{-1}\,\sin(\theta+\sin^2\theta)$$

$$\therefore\;S=e^{\sin\theta\cos\theta}.\,\sin(\theta+\sin^2\theta).$$

Summation by Logarithmic Series.

24. $S=x\cos\theta-\frac{x^2}{2}\cos 2\theta+\frac{x^3}{3}\cos 3\theta-\frac{x^4}{4}\cos 4\theta+\ldots$ to ∞.

Let $2\cos\theta=y+\frac{1}{y}$,

$$2S=x\left(y+\frac{1}{y}\right)-\frac{x^2}{2}\left(y^2+\frac{1}{y^2}\right)+\frac{x^3}{3}\left(y^3+\frac{1}{y^3}\right)-\ldots$$

$$=xy-\frac{x^2y^2}{2}+\frac{x^3y^3}{3}-\ldots+\left\{\frac{x}{y}-\frac{x^2}{2y^2}+\frac{x^3}{3y^3}-\ldots\right\}$$

$$=\log(1+xy)+\log\left(1+\frac{x}{y}\right)$$

$$=\log(1+xy)\left(1+\frac{x}{y}\right)$$

$$=\log\left(1+x\left(y+\frac{1}{y}\right)+x^2\right)$$

$$=\log(1+2x\cos\theta+x^2).$$

25. $S = x \sin\theta - \frac{x^2}{2}\sin 2\theta + \frac{x^3}{3}\sin 3\theta - \frac{x^4}{4}\sin 4\theta + \ldots$ to ∞.

$$2\sqrt{-1}S = x\left(y - \frac{1}{y}\right) - \frac{x^2}{2}\left(y^2 - \frac{1}{y^2}\right) + \frac{x^3}{3}\left(y^3 - \frac{1}{y^3}\right) - \ldots$$

$$= xy - \frac{x^2y^2}{2} + \frac{x^3y^3}{3} - \ldots$$

$$- \left\{\frac{x}{y} - \frac{x^2}{2y^2} + \frac{x^3}{3y^3} - \ldots\right\}$$

$$= \log(1 + xy) - \log\left(1 + \frac{x}{y}\right)$$

$$= \log\frac{1 + xy}{1 + \frac{x}{y}}$$

$$\therefore\ e^{2\sqrt{-1}S} = \frac{1 + xy}{1 + \frac{x}{y}}$$

$$\frac{e^{2\sqrt{-1}S} - 1}{e^{2\sqrt{-1}S} + 1} = \frac{x\left(y - \frac{1}{y}\right)}{2 + x\left(y + \frac{1}{y}\right)} = \frac{2\sqrt{-1}\,x\sin\theta}{2 + 2x\cos\theta}$$

$$\sqrt{-1}\tan S = \frac{\sqrt{-1}\,x\sin\theta}{1 + x\cos\theta}$$

$$S = \tan^{-1}\frac{x\sin\theta}{1 + x\cos\theta}.$$

26. $S = \cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \ldots$ to ∞.

Write $x = 1$ in (24),

$$2S = \log(2 + 2\cos\theta) \qquad \text{(lxxxvii. 5.)}$$

$$= \log\left(4\cos^2\frac{\theta}{2}\right)$$

$$= \log\left(2\cos\frac{\theta}{2}\right)^2$$

$$= 2\log\left(2\cos\frac{\theta}{2}\right)$$

$$\therefore\ S = \log\left(2\cos\frac{\theta}{2}\right).$$

27. $S=\sin\theta\,\dfrac{\sin\theta}{1}-\sin 2\theta\,\dfrac{\sin^2\theta}{2}+\sin 3\theta\,\dfrac{\sin^3\theta}{3}-\ldots$ to ∞.

Write $x=\sin\theta$ in (25),

$$S=\tan^{-1}\frac{\sin^2\theta}{1+\sin\theta\cos\theta}$$

$$=\cot^{-1}\frac{\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta}{\sin^2\theta}$$

$$=\cot^{-1}(1+\cot^2\theta+\cot\theta).$$

Summation by G. P.

28. $S=1+x\cos\theta+x^2\cos 2\theta+x^3\cos 3\theta+\ldots$ to ∞.

Let $2\cos\theta=z+\dfrac{1}{z}$,

$$2S=2+x\left(z+\frac{1}{z}\right)+x^2\left(z^2+\frac{1}{z^2}\right)+x^3\left(z^3+\frac{1}{z^3}\right)+\cdots$$

$$=1+xz+x^2z^2+\ldots+1+\frac{x}{z}+\frac{x^2}{z^2}+\cdots$$

$$=\frac{1}{1-xz}+\frac{1}{1-\dfrac{x}{z}}$$

$$=\frac{2-x\left(z+\dfrac{1}{z}\right)}{1-x\left(z+\dfrac{1}{z}\right)+x^2}$$

$$=\frac{2-2x\cos\theta}{1-2x\cos\theta+x^2}$$

$$\therefore\ S=\frac{1-x\cos\theta}{1-2x\cos\theta+x^2}.$$

29. PROP. $S = \sin a + x \sin(a+\beta) + x^2 \sin(a+2\beta) + \dots$ to n terms.

If $y = \cos a + \sqrt{-1}\sin a, \quad z = \cos\beta + \sqrt{-1}\sin\beta.$

$$2\sqrt{-1}S = \left(y - \frac{1}{y}\right) + x\left(yz - \frac{1}{yz}\right) + x^2\left(yz^2 - \frac{1}{yz^2}\right) + \cdots$$

$$= y\,(1 + xz + x^2z^2 + \dots) - \frac{1}{y}\left(1 + \frac{x}{z} + \frac{x^2}{z^2} + \cdots\right)$$

$$= y\,\frac{x^nz^n - 1}{xz - 1} - \frac{1}{y}\,\frac{\frac{x^n}{z^n} - 1}{\frac{x}{z} - 1} \qquad \text{(xcviii. 8.)}$$

$$= y\,\frac{x^nz^n - 1}{xz - 1} - \frac{y^{-1}(x^nz^{-n} - 1)}{xz^{-1} - 1}$$

$$= \frac{x^{n+1}(yz^{n-1} - y^{-1}z^{-(n-1)}) - x^n(yz^n - y^{-1}z^{-n}) - x(yz^{-1} - y^{-1}z) + y - y^{-1}}{x^2 - x(z + z^{-1}) + 1}$$

$$\therefore S_n = \frac{x^{n+1}\sin(a + (n-1)\beta) - x^n \sin(a + n\beta) - x\sin(a - \beta) + \sin a}{1 - 2x\cos\beta + x^2}.$$

If x be less than unity, and n be indefinitely increased,

$$S_\infty = \frac{\sin a - x\sin(a - \beta)}{1 - 2x\cos\beta + x^2}.$$

Similarly we should find

$$\cos a + x\cos(a+\beta) + x^2\cos(a+2\beta) + \dots \text{ to } n \text{ terms} \qquad \text{(cvii. 5.)}$$

$$= \frac{x^{n+1}\cos(a + (n-1)\beta) - x^n\cos(a + n\beta) - x\cos(a - \beta) + \cos a}{1 - 2x\cos\beta + x^2}.$$

$$\text{and} \quad S_\infty = \frac{\cos a - x\cos(a - \beta)}{1 - 2x\cos\beta + x^2}.$$

Summation by Binomial Theorem.

30. **Prop.** $C=\cos a+x\ \cos(a+\beta)+\frac{x(x-1)}{\underline{|2}}\cos(a+2\beta)+\ .\ .\ .$ to ∞,

$$S=\sin a+x\ \sin(a+\beta)+\frac{x(x-1)}{\underline{|2}}\sin(a+2\beta)+\ .\ .\ .\ \text{to } \infty.$$

$$\therefore\ C+\sqrt{-1}\,S=e^{\alpha\sqrt{-1}}+xe^{(\alpha+\beta)\sqrt{-1}}+\frac{x(x-1)}{\underline{|2}}e^{(\alpha+2\beta)\sqrt{-1}}+\ .\ .\ .$$

$$=e^{\alpha\sqrt{-1}}\left(1+xe^{\beta\sqrt{-1}}+\frac{x(x-1)}{\underline{|2}}e^{2\beta\sqrt{-1}}+\ .\ .\ .\right)$$

$$=e^{\alpha\sqrt{-1}}\ (1+e^{\beta\sqrt{-1}})^x$$

$$=e^{\alpha\sqrt{-1}}\left\{e^{\frac{\beta\sqrt{-1}}{2}}\left(e^{\frac{\beta\sqrt{-1}}{2}}+e^{\frac{-\beta\sqrt{-1}}{2}}\right)\right\}^x$$

$$=e^{\alpha\sqrt{-1}}\,.\,e^{\frac{x\beta\sqrt{-1}}{2}}\left(2\cos\frac{\beta}{2}\right)^x$$

$$=2^x\cos^x\frac{\beta}{2}\,e^{\left(\alpha+\frac{x\beta}{2}\right)\sqrt{-1}}$$

$$=2^x\cos^x\frac{\beta}{2}\left\{\cos\left(a+\frac{x\beta}{2}\right)+\sqrt{-1}\ \sin\left(a+\frac{x\beta}{2}\right)\right\}$$

Equate possible and impossible parts,

$$\therefore\ C=2^x\cos^x\frac{\beta}{2}\cos\left(a+\frac{x\beta}{2}\right).$$

$$S=2^x\cos^x\frac{\beta}{2}\sin\left(a+\frac{x\beta}{2}\right).$$

31. $S=1+2a\cos\theta+3a^2\cos 2\theta+4a^3\cos^3\theta+\ .\ .\ .$ to ∞.

$$2\,S=2+2a\left(x+\frac{1}{x}\right)+3a^2\left(x^2+\frac{1}{x^2}\right)+4a^3\ \left.x^3+\frac{1}{x^3}\right)+\ .\ .\ .$$

$$=1+2ax+3a^2x^2+\ .\ .\ .$$

$$+1+\frac{2a}{x}+\frac{3a^2}{x^2}+\ .\ .\ .$$

$$=(1-ax)^{-2}+\left(1-\frac{a}{x}\right)^{-2}$$

$$=\frac{2-2a\left(x+\frac{1}{x}\right)+a^2\left(x^2+\frac{1}{x^2}\right)}{\left\{1+a^2-a\left(x+\frac{1}{x}\right)\right\}^2}\qquad\text{(cxx. 4.)}$$

$$=\frac{2-4a\cos\theta+2a^2\cos 2\theta}{(1+a^2-2a\cos\theta)^2}$$

$$\therefore\ S=\frac{1-2a\cos\theta+a^2\cos 2\theta}{(1+a^2-2a\cos\theta)^2}.$$

29. Prop. $S = \sin a + x \sin (a+\beta) + x^2 \sin (a+2\beta) + \dots$ to n terms.

If $y = \cos a + \sqrt{-1} \sin a$, $\quad z = \cos \beta + \sqrt{-1} \sin \beta$.

$$2\sqrt{-1}S = \left(y - \frac{1}{y}\right) + x\left(yz - \frac{1}{yz}\right) + x^2\left(yz^2 - \frac{1}{yz^2}\right) + \dots$$

$$= y\,(1 + xz + x^2z^2 + \dots) - \frac{1}{y}\left(1 + \frac{x}{z} + \frac{x^2}{z^2} + \dots\right)$$

$$= y\,\frac{x^nz^n - 1}{xz - 1} - \frac{1}{y}\,\frac{\dfrac{x^n}{z^n} - 1}{\dfrac{x}{z} - 1} \qquad \text{(xcviii. 8.)}$$

$$= y\,\frac{x^nz^n - 1}{xz - 1} - \frac{y^{-1}(x^nz^{-n} - 1)}{xz^{-1} - 1}$$

$$= \frac{x^{n+1}(yz^{n-1} - y^{-1}z^{-(n-1)}) - x^n(yz^n - y^{-1}z^{-n}) - x(yz^{-1} - y^{-1}z) + y - y^{-1}}{x^2 - x(z + z^{-1}) + 1}$$

$$\therefore S_n = \frac{x^{n+1} \sin (a + (n-1)\beta) - x^n \sin (a + n\beta) - x \sin (a - \beta) + \sin a}{1 - 2x \cos \beta + x^2}.$$

If x be less than unity, and n be indefinitely increased,

$$S_\infty = \frac{\sin a - x \sin (a - \beta)}{1 - 2x \cos \beta + x^2}.$$

Similarly we should find

$$\cos a + x \cos (a + \beta) + x^2 \cos (a + 2\beta) +$$

$$= \frac{x^{n+1} \cos (a + (n-1)\beta) - x^n \cos (a + n}{1 - 2x \cos \beta}$$

and $\quad S_\infty = \frac{\text{co}}{}$

Summation by Binomial Theorem.

30. PROP. $C=\cos\alpha+x\cos(\alpha+\beta)+\frac{x(x-1)}{\underline{|2}}\cos(\alpha+2\beta)+\ldots$ to ∞,

$$S=\sin\alpha+x\sin(\alpha+\beta)+\frac{x(x-1)}{\underline{|2}}\sin(\alpha+2\beta)+\ldots \text{ to } \infty.$$

$$\therefore\ C+\sqrt{-1}\,S=e^{\alpha\sqrt{-1}}+xe^{(\alpha+\beta)\sqrt{-1}}+\frac{x(x-1)}{\underline{|2}}e^{(\alpha+2\beta)\sqrt{-1}}+\ldots$$

$$=e^{\alpha\sqrt{-1}}\left(1+xe^{\beta\sqrt{-1}}+\frac{x(x-1)}{\underline{|2}}e^{2\beta\sqrt{-1}}+\ldots\right)$$

$$=e^{\alpha\sqrt{-1}}\left(1+e^{\beta\sqrt{-1}}\right)^x$$

$$=e^{\alpha\sqrt{-1}}\left\{e^{\frac{\beta\sqrt{-1}}{2}}\left(e^{\frac{\beta\sqrt{-1}}{2}}+e^{\frac{-\beta\sqrt{-1}}{2}}\right)\right\}^x$$

$$=e^{\alpha\sqrt{-1}}\,.\,e^{\frac{x\beta\sqrt{-1}}{2}}\left(2\cos\frac{\beta}{2}\right)^x$$

$$=2^x\cos^x\frac{\beta}{2}\,e^{\left(\alpha+\frac{x\beta}{2}\right)\sqrt{-1}}$$

$$=2^x\cos^x\frac{\beta}{2}\left\{\cos\left(\alpha+\frac{x\beta}{2}\right)+\sqrt{-1}\sin\left(\alpha+\frac{x\beta}{2}\right)\right\}$$

Equate possible and impossible parts,

$$\therefore\ C=2^x\cos^x\frac{\beta}{2}\cos\left(\alpha+\frac{x\beta}{2}\right).$$

$$S=2^x\cos^x\frac{\beta}{2}\sin\left(\alpha+\frac{x\beta}{2}\right).$$

31. $S=1+2a\cos\theta+3a^2\cos2\theta+4a^3\cos^3\theta+\ldots$ to ∞.

$$2\,S=2+2a\left(x+\frac{1}{x}\right)+3a^2\left(x^2+\frac{1}{x^2}\right)+4a^3\ x^3+\frac{1}{x^3}\Big)+\ldots$$

$$=1+2ax+3a^2x^2+\ldots$$

$$\frac{2a}{x}+3\ldots$$

[illegible]

$$\ldots+\frac{1}{x^2}\Big) \qquad \text{(cxx. 4.)}$$

$$\ldots\Big)\Big\}^2$$

$$\ldots s\,2\theta \,/\, \ldots)^2$$

$$\ldots s\,2\theta \,/\, \ldots\theta)^2.$$

XXVIII.

Application of Trigonometry to the Solution of Equations.

1. Prop. Solve $ax^2-bx+c=0$.

$$x=\frac{b\pm\sqrt{b^2-4ac}}{2a}=\frac{b}{2a}\left\{1\pm\sqrt{1-\frac{4ac}{b^2}}\right\}$$

(i.) When x is possible, $b^2>4ac$; let $\frac{4ac}{b^2}=\sin^2\theta$,

$$\therefore x=\frac{b}{2a}\{1\pm\cos\theta\}=\frac{b}{a}\cos^2\frac{\theta}{2}\quad\text{or }\frac{b}{a}\sin^2\frac{\theta}{2}.$$

(ii.) If $b^2<4ac$, the roots are impossible; let $\frac{4ac}{b^2}=\sec^2\theta$,

$$\text{Then } x=\frac{b}{2a}\{1\pm\sqrt{1-\sec^2\theta}\}=\frac{b}{2a}\{1\pm\sqrt{-1}\tan\theta\}.$$

2. Prop. Solve $ax^2-bx+c=0$.

$$x=\frac{b\pm\sqrt{b^2+4ac}}{2a}=\frac{b}{2a}\left\{1\pm\sqrt{1+\frac{4ac}{b^2}}\right\}$$

Let $\frac{4ac}{b^2}=\tan^2\theta$

$$\therefore x=\frac{b}{2a}(1\pm\sec\theta)=\frac{b}{a}\cos^2\frac{\theta}{2}\sec\theta\quad\text{or }-\frac{b}{a}\sin^2\frac{\theta}{2}\sec\theta.$$

When a, b, c are large numerical quantities, it would be laborious to calculate the value of x by the ordinary method. Here θ and x are in forms adapted to logarithmic computation, and may in all cases be determined readily.

3. **Prop.** Solve $x^3-qx-r=0$; since any cubic equation may be transformed into another in which the second term is wanting.

$$\text{Let } x=y\cos\theta,$$

$$y^3\cos^3\theta-qy\cos\theta-r=0 \qquad \text{(cviii. 1.)}$$

$$\cos^3\theta-\frac{q}{y^2}\cos\theta-\frac{r}{y^3}=0,$$

$$\text{and } \cos 3\theta=4\cos^3\theta-3\cos\theta$$

$$\text{or } \cos^3\theta-\frac{3}{4}\cos\theta-\frac{1}{4}\cos 3\theta=0.$$

Comparing the equations,

$$\frac{q}{y^2}=\frac{3}{4}, \quad \therefore\ y=2\sqrt{\frac{q}{3}},$$

$$\frac{r}{y^3}=\frac{\cos 3\theta}{4}, \quad \therefore\ \cos 3\theta=\frac{4r}{y^3}=\frac{r}{2}\sqrt{\frac{27}{q^3}},$$

which gives 3θ, and thus θ is known.

And $x=y\cos\theta$,

$$=2\sqrt{\frac{q}{3}}\cos\theta,\ 2\sqrt{\frac{q}{3}}\cos\frac{2\pi+3\theta}{3},\ 2\sqrt{\frac{q}{3}}\cos\frac{2\pi-3\theta}{3}$$

Since $\cos 3\theta<1$, $\frac{r}{2}\sqrt{\frac{27}{q^3}}<1$,

$$\text{or } \frac{r^2}{4}<\frac{q^3}{27},$$

$\therefore\ \frac{r^2}{4}-\frac{q^3}{27}$ cannot be positive, which corresponds with the *irreducible* case of *Cardan's Solution* of a cubic equation, all the roots being real. This solution is therefore available whenever Cardan's Solution fails.

Ex. 1. $x^3-3x-1=0$.

$$y=2, \quad \cos 3\theta=\frac{1}{2}, \quad 3\theta=60^\circ \text{ or } \theta=20^\circ,$$

$\therefore$ the solutions are $2\cos 20^\circ$, $-2\cos 40^\circ$, $-2\cos 80^\circ$.

Ex. 2. $x^3-6x-4=0$.

$$y=2\sqrt{2}, \quad \cos 3\theta=\frac{1}{\sqrt{2}}, \quad 3\theta=45^\circ \text{ or } \theta=15^\circ,$$

$\therefore$ the solutions are $2\sqrt{2}\cos 15^\circ$, $-2\sqrt{2}\cos 45^\circ$, $-2\sqrt{2}\cos 75^\circ$.

4. **Prop.** Solve $x^3-qx+r=0$.

$$\text{Let } x=y\sin\theta,$$

$$y^3\sin^3\theta-qy\sin\theta+r=0,$$

$$\left.\begin{aligned}\sin^3\theta-\frac{q}{y^2}\sin\theta+\frac{r}{y^3}&=0\\ \text{But }\sin^3\theta-\frac{3}{4}\sin\theta+\frac{1}{4}\sin 3\theta&=0\end{aligned}\right\}$$

Comparing the equations,

$$\frac{q}{y^2}=\frac{3}{4}, \quad y=2\sqrt{\frac{q}{3}},$$

$$\frac{r}{y^3}=\frac{\sin 3\theta}{4}, \quad \sin 3\theta=\frac{4r}{y^3}=\frac{r}{2}\sqrt{\frac{27}{q^3}},$$

which gives 3θ, and thus θ is known.

And $x=y\sin\theta$,

$$=2\sqrt{\frac{q}{3}}\sin\theta, \quad 2\sqrt{\frac{q}{3}}\sin\frac{2\pi+3\theta}{3}, \quad -2\sqrt{\frac{q}{3}}\sin\frac{2\pi-3\theta}{3}.$$

Since $\sin 3\theta<1$, $\therefore \frac{r^2}{4}<\frac{q^3}{27}$,

$\therefore$ the equation can be solved by this method, only when all the roots are real.

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Comparing the equations,

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$$\frac{r}{y^3}=\frac{\sin 3\theta}{4},\quad \sin 3\theta=\frac{4r}{y^3}=\frac{r}{2}\sqrt{\frac{27}{q^3}},$$

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EXAMINATION PAPERS

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$$\text{Let } x=y\sin\theta,$$

$$y^3\sin^3\theta-qy\sin\theta+r=0,$$

$$\left.\begin{array}{r}\sin^3\theta-\frac{q}{y^2}\sin\theta+\frac{r}{y^3}=0\\ \text{But }\sin^3\theta-\frac{3}{4}\sin\theta+\frac{1}{4}\sin 3\theta=0\end{array}\right\}$$

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$$\frac{q}{y^2}=\frac{3}{4},\quad y=2\sqrt{\frac{q}{3}},$$

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EXAMINATION PAPERS

21. $\text{Tan}^{-1}\dfrac{1}{1+1+1^2}+\tan^{-1}\dfrac{1}{1+2+2^2}+\ldots$ to n terms.

$$\tan^{-1}\frac{1}{1+1+1^2}=\tan^{-1}\frac{1}{1}-\tan^{-1}\frac{1}{2}$$

$$\tan^{-1}\frac{1}{1+2+2^2}=\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{3}$$

$$\text{etc.}=\text{etc.}$$

$$\therefore S_n=\tan^{-1}1-\tan^{-1}\frac{1}{n+1}$$

$$=\frac{\pi}{4}-\tan^{-1}\frac{1}{n+1}.$$

Summation by Exponential Theorem.

22. **Prop.** Let $C=x\cos(\alpha+\beta)+\dfrac{x^2}{\underline{|2}}\cos(\alpha+2\beta)$

$$+\frac{x^3}{\underline{|3}}\cos(\alpha+3\beta)+\ldots \text{ to } \infty.$$

$$S=x\sin(\alpha+\beta)+\frac{x^2}{\underline{|2}}\sin(\alpha+2\beta)$$

$$+\frac{x^3}{\underline{|3}}\sin(\alpha+3\beta)+\ldots \text{ to } \infty.$$

$$C+\sqrt{-1}\,S=xe^{(\alpha+\beta)\sqrt{-1}}+\frac{x^2}{\underline{|2}}e^{(\alpha+2\beta)\sqrt{-1}}+\frac{x^3}{\underline{|3}}e^{(\alpha+3\beta)\sqrt{-1}}+\ldots$$

$$=e^{\alpha\sqrt{-1}}\left\{xe^{\beta\sqrt{-1}}+\frac{x^2}{\underline{|2}}e^{2\beta\sqrt{-1}}+\frac{x^3}{\underline{|3}}e^{3\beta\sqrt{-1}}+\ldots\right\}$$

$$=e^{\alpha\sqrt{-1}}\left\{e^{xe^{\beta\sqrt{-1}}}-1\right\}$$

$$=e^{\alpha\sqrt{-1}}\left\{e^{x(\cos\beta+\sqrt{-1}\sin\beta)}-1\right\}$$

$$[illegible]\,e^{x\cos\beta}\,e^{(\alpha+x\sin\beta)\sqrt{-1}}-e^{\alpha\sqrt{-1}}$$

$$[illegible]\,\beta\left\{\cos(\alpha+x\sin\beta)+\sqrt{-1}\sin(\alpha+x\sin\beta)\right\}$$

$\frac{1}{2}$ (tan α + [illegible] sin α)

[illegible] cot $\frac{\pi}{2n-1}$ [illegible] α.

[illegible] sin α.

$$23.\ \mathrm{S}=\sin\theta+\frac{\sin\theta}{1}\sin 2\theta+\frac{\sin^2\theta}{1.2}\sin 3\theta+\ .\ .\ .\ \text{to }\infty.$$

$$2\sqrt{-1}\ S=\left(x-\frac{1}{x}\right)+\sin\theta\left(x^2-\frac{1}{x^2}\right)+\frac{\sin^2\theta}{1.2}\left(x^3-\frac{1}{x^3}\right)+\ .\ .\ .$$

$$=x\left(1+x\sin\theta+\frac{x^2\sin 2\theta}{1.2}+\ldots\right)-\frac{1}{x}\left(1+\frac{\sin\theta}{x}+\frac{\sin^2\theta}{1.2x^2}+\ldots\right)$$

$$=x\ e^{x\sin\theta}-\frac{1}{x}\ e^{\frac{\sin\theta}{x}}$$

$$=x\ e^{\sin\theta\,(\cos\theta+\sqrt{-1}\sin\theta)}-\frac{1}{x}\ e^{\sin\theta\ \cos\theta-\sqrt{-1}\sin\theta}$$

$$=e^{\sin\theta\cos\theta}\left[xe^{\sqrt{-1}\sin^2\theta}-\frac{1}{x}e^{-\sqrt{-1}\sin^2\theta}\right]$$

$$=e^{\sin\theta\cos\theta}[\,e^{\theta\sqrt{-1}}.\ e^{\sqrt{-1}\sin^2\theta}-e^{-\theta\sqrt{-1}}.\ e^{-\sqrt{-1}\sin^2\theta}\,]$$

$$=e^{\sin\theta\cos\theta}[\,e^{\sqrt{-1}(\theta+\sin^2\theta)}-e^{-\sqrt{-1}(\theta+\sin^2\theta)}\,]$$

$$=e^{\sin\theta\cos\theta}.\ 2\sqrt{-1}\ \sin(\theta+\sin^2\theta)$$

$$\therefore\ S=e^{\sin\theta\cos\theta}.\ \sin(\theta+\sin^2\theta).$$

Summation by Logarithmic Series.

$$24.\ S=x\cos\theta-\frac{x^2}{2}\cos 2\theta+\frac{x^3}{3}\cos 3\theta-\frac{x^4}{4}\cos 4\theta+\ .\ .\ .\ \text{to }\infty.$$

Let $2\cos\theta=y+\dfrac{1}{y}$,

$$2S=x\left(y+\frac{1}{y}\right)-\frac{x^2}{2}\left(y^2+\frac{1}{y^2}\right)+\frac{x^3}{3}\left(y^3+\frac{1}{y^3}\right)-\ .\ .\ .$$

$$=xy-\frac{x^2y^2}{2}+\frac{x^3y^3}{3}-\ .\ .\ .+\left\{\frac{x}{y}-\frac{x^2}{2y^2}+\frac{x^3}{3y^3}-\ .\ .\ .\right\}$$

$$=\log(1+xy)+\log\left(1+\frac{x}{y}\right)$$

$$=\log(1+xy)\left(1+\frac{x}{y}\right)$$

$$=\log\left(1+x\left(y+\frac{1}{y}\right)+x^2\right)$$

$$=\log(1+2x\cos\theta+x^2).$$

25. $S = x\sin\theta - \frac{x^2}{2}\sin 2\theta + \frac{x^3}{3}\sin 3\theta - \frac{x^4}{4}\sin 4\theta + \ldots$ to ∞.

$$2\sqrt{-1}S = x\left(y - \frac{1}{y}\right) - \frac{x^2}{2}\left(y^2 - \frac{1}{y^2}\right) + \frac{x^3}{3}\left(y^3 - \frac{1}{y^3}\right) - \ldots$$

$$= xy - \frac{x^2y^2}{2} + \frac{x^3y^3}{3} - \ldots$$

$$- \left\{\frac{x}{y} - \frac{x^2}{2y^2} + \frac{x^3}{3y^3} - \ldots\right\}$$

$$= \log(1 + xy) - \log\left(1 + \frac{x}{y}\right)$$

$$= \log\frac{1 + xy}{1 + \frac{x}{y}}$$

$$\therefore\ e^{2\sqrt{-1}S} = \frac{1 + xy}{1 + \frac{x}{y}}$$

$$\frac{e^{2\sqrt{-1}S} - 1}{e^{2\sqrt{-1}S} + 1} = \frac{x\left(y - \frac{1}{y}\right)}{2 + x\left(y + \frac{1}{y}\right)} = \frac{2\sqrt{-1}\,x\sin\theta}{2 + 2x\cos\theta}$$

$$\sqrt{-1}\tan S = \frac{\sqrt{-1}\,x\sin\theta}{1 + x\cos\theta}$$

$$S = \tan^{-1}\frac{x\sin\theta}{1 + x\cos\theta}.$$

26. $S = \cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \ldots$ to ∞.

Write $x=1$ in (24),

$$2S = \log(2 + 2\cos\theta) \qquad \text{(lxxxvii. 5.)}$$

$$= \log\left(4\cos^2\frac{\theta}{2}\right)$$

$$= \log\left(2\cos\frac{\theta}{2}\right)^2$$

$$= 2\log\left(2\cos\frac{\theta}{2}\right)$$

$$\therefore\ S = \log\left(2\cos\frac{\theta}{2}\right).$$

27. $S=\sin\theta\,\dfrac{\sin\theta}{1}-\sin 2\theta\,\dfrac{\sin^2\theta}{2}+\sin 3\theta\,\dfrac{\sin^3\theta}{3}-\dots$ to ∞.

Write $x=\sin\theta$ in (25),

$$S=\tan^{-1}\frac{\sin^2\theta}{1+\sin\theta\cos\theta}$$

$$=\cot^{-1}\frac{\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta}{\sin^2\theta}$$

$$=\cot^{-1}(1+\cot^2\theta+\cot\theta).$$

Summation by G. P.

28. $S=1+x\cos\theta+x^2\cos 2\theta+x^3\cos 3\theta+\dots$ to ∞.

Let $2\cos\theta=z+\dfrac{1}{z}$,

$$2S=2+x\left(z+\frac{1}{z}\right)+x^2\left(z^2+\frac{1}{z^2}\right)+x^3\left(z^3+\frac{1}{z^3}\right)+\dots$$

$$=1+xz+x^2z^2+\dots+1+\frac{x}{z}+\frac{x^2}{z^2}+\dots$$

$$=\frac{1}{1-xz}+\frac{1}{1-\dfrac{x}{z}}$$

$$=\frac{2-x\left(z+\dfrac{1}{z}\right)}{1-x\left(z+\dfrac{1}{z}\right)+x^2}$$

$$=\frac{2-2x\cos\theta}{1-2x\cos\theta+x^2}$$

$$\therefore\ S=\frac{1-x\cos\theta}{1-2x\cos\theta+x^2}.$$

29. **Prop.** $S=\sin\alpha+x\sin(\alpha+\beta)+x^2\sin(\alpha+2\beta)+\dots$ to n terms.

If $y=\cos\alpha+\sqrt{-1}\sin\alpha$, $\quad z=\cos\beta+\sqrt{-1}\sin\beta$.

$$2\sqrt{-1}S=\left(y-\frac{1}{y}\right)+x\left(yz-\frac{1}{yz}\right)+x^2\left(yz^2-\frac{1}{yz^2}\right)+\dots$$

$$=y\,(1+xz+x^2z^2+\dots)-\frac{1}{y}\left(1+\frac{x}{z}+\frac{x^2}{z^2}+\dots\right)$$

$$=y\,\frac{x^nz^n-1}{xz-1}-\frac{1}{y}\,\frac{\frac{x^n}{z^n}-1}{\frac{x}{z}-1} \qquad \text{(xcviii. 8.)}$$

$$=y\,\frac{x^nz^n-1}{xz-1}-\frac{y^{-1}(x^nz^{-n}-1)}{xz^{-1}-1}$$

$$=\frac{x^{n+1}(yz^{n-1}-y^{-1}z^{-(n-1)})-x^n(yz^n-y^{-1}z^{-n})-x(yz^{-1}-y^{-1}z)+y-y^{-1}}{x^2-x(z+z^{-1})+1}$$

$$\therefore\ S_n=\frac{x^{n+1}\sin(\alpha+(n-1)\beta)-x^n\sin(\alpha+n\beta)-x\sin(\alpha-\beta)+\sin\alpha}{1-2x\cos\beta+x^2}.$$

If x be less than unity, and n be indefinitely increased,

$$S_\infty=\frac{\sin\alpha-x\sin(\alpha-\beta)}{1-2x\cos\beta+x^2}.$$

Similarly we should find

$$\cos\alpha+x\cos(\alpha+\beta)+x^2\cos(\alpha+2\beta)+\dots \text{ to } n \text{ terms} \qquad \text{(cvii. 5.)}$$

$$=\frac{x^{n+1}\cos(\alpha+(n-1)\beta)-x^n\cos(\alpha+n\beta)-x\cos(\alpha-\beta)+\cos\alpha}{1-2x\cos\beta+x^2}.$$

and $\quad S_\infty=\dfrac{\cos\alpha-x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2}.$

Summation by Binomial Theorem.

30. PROP. $C=\cos\alpha+x\cos(\alpha+\beta)+\frac{x(x-1)}{\underline{|2}}\cos(\alpha+2\beta)+\ .\ .\ .$ to ∞,

$S=\sin\alpha+x\sin(\alpha+\beta)+\frac{x(x-1)}{\underline{|2}}\sin(\alpha+2\beta)+\ .\ .\ .$ to ∞.

$$\therefore\ C+\sqrt{-1}\,S=e^{\alpha\sqrt{-1}}+xe^{(\alpha+\beta)\sqrt{-1}}+\frac{x(x-1)}{\underline{|2}}e^{(\alpha+2\beta)\sqrt{-1}}+\ .\ .\ .$$

$$=e^{\alpha\sqrt{-1}}\left(1+xe^{\beta\sqrt{-1}}+\frac{x(x-1)}{\underline{|2}}e^{2\beta\sqrt{-1}}+\ .\ .\ .\right)$$

$$=e^{\alpha\sqrt{-1}}\left(1+e^{\beta\sqrt{-1}}\right)^x$$

$$=e^{\alpha\sqrt{-1}}\left\{e^{\frac{\beta\sqrt{-1}}{2}}\left(e^{\frac{\beta\sqrt{-1}}{2}}+e^{\frac{-\beta\sqrt{-1}}{2}}\right)\right\}^x$$

$$=e^{\alpha\sqrt{-1}}\,.\,e^{\frac{x\beta\sqrt{-1}}{2}}\left(2\cos\frac{\beta}{2}\right)^x$$

$$=2^x\cos^x\frac{\beta}{2}\,e^{\left(\alpha+\frac{x\beta}{2}\right)\sqrt{-1}}$$

$$=2^x\cos^x\frac{\beta}{2}\left\{\cos\left(\alpha+\frac{x\beta}{2}\right)+\sqrt{-1}\sin\left(\alpha+\frac{x\beta}{2}\right)\right\}$$

Equate possible and impossible parts,

$$\therefore\ C=2^x\cos^x\frac{\beta}{2}\cos\left(\alpha+\frac{x\beta}{2}\right).$$

$$S=2^x\cos^x\frac{\beta}{2}\sin\left(\alpha+\frac{x\beta}{2}\right).$$

31. $S=1+2a\cos\theta+3a^2\cos 2\theta+4a^3\cos^3\theta+\ .\ .\ .$ to ∞.

$$2\,S=2+2a\left(x+\frac{1}{x}\right)+3a^2\left(x^2+\frac{1}{x^2}\right)+4a^3\ \ x^3+\frac{1}{x^3}\Big)+\ .\ .\ .$$

$$=1+2ax+3a^2x^2+\ .\ .\ .$$

$$+1+\frac{2a}{x}+\frac{3a^2}{x^2}+\ .\ .\ .$$

$$=(1-ax)^{-2}+\left(1-\frac{a}{x}\right)^{-2}$$

$$=\frac{2-2a\left(x+\frac{1}{x}\right)+a^2\left(x^2+\frac{1}{x^2}\right)}{\left\{1+a^2-a\left(x+\frac{1}{x}\right)\right\}^2}\qquad\text{(cxx. 4.)}$$

$$=\frac{2-4a\cos\theta+2a^2\cos 2\theta}{(1+a^2-2a\cos\theta)^2}$$

$$\therefore\ S=\frac{1-2a\cos\theta+a^2\cos 2\theta}{(1+a^2-2a\cos\theta)^2}.$$

VIII.

1. Name the trigonometrical ratios, and define them.

Prove that $\sec^2 A + \text{cosec}^2 A = \sec^2 A \ \text{cosec}^2 A$.

2. Show that—

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}; \text{ and that}$$

$$* \frac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} - \frac{\cos A - \cos 3A}{\sin 3A - \sin A} = \frac{\sin A}{\cos 2A \cos 3A}.$$

3. Find the value of sin 15°, and prove that—

$$\sin 3(A - 15°) = 4 \cos(A - 45°) \cos(A + 15°) \sin(A - 15°).$$

4. Show that the sines of the angles of a triangle are proportional to the opposite sides.

* If α, β, γ are the perpendiculars from the angular points of a triangle upon the opposite sides, prove that—

$$a \sin A + b \sin B + c \sin C = 2(\alpha \cos A + \beta \cos B + \gamma \cos C).$$

* 5. The sides BC, CA, AB of a triangle are as 4 : 5 : 6. Find the angle B.

$$\text{L} \cos 27° \ 53' = 9 \cdot 9464040.$$
$$\text{L} \cos 27° \ 54' = 9 \cdot 9463371.$$
$$\log 2 = \cdot 3010300.$$

6. In a triangle of given perimeter, prove that the radii of the escribed circles are proportional to the tangents of the semi-angles opposite to the sides to which they are escribed. Show that the area of the triangle formed by joining the centres of the escribed circles of the original triangle $= \frac{s.a}{\sin A}$, where s is the semi-perimeter of the original triangle.

* 7. The angular elevation of a steeple at a place due south of it is 45°, and at another place due west of the former station the elevation is 15°; show that the height of the steeple is $\frac{a}{2}(3^{\frac{1}{4}} - 3^{-\frac{1}{4}})$, a being the distance between the places.

6. Given $\left.\begin{matrix} b=8{\cdot}4 \\ c=12 \end{matrix}\right\}$ inches, $B=37^\circ\ 36'$, find A.

$$\log 7 = {\cdot}845098.$$
$$\text{L} \sin 37^\circ\ 36' = 9{\cdot}7854332.$$
$$\text{L} \sin 60^\circ\ 39' = 9{\cdot}9403381.$$
$$\text{diff. } 1' = 711.$$

I.*

1. What is meant by the circular measure of an angle? Prove the formula $\theta = \frac{\text{arc}}{\text{radius}}$. What would this formula become if the unit of circular measure was 36°?

* 2. Prove from a figure that $\cos(A-B) = \cos A \ \cos B + \sin A \ \sin B$, when A lies between 315° and 360°, and $A-B$ between 180° and 225°.

3. Solve the following equations, giving the general expressions for the angles—

(i.) $\cos\theta = 0$.
(ii.) $\sin\theta + \cos\theta = 1$.
(iii.) $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta \,.\, \tan 4\theta \,.\, \tan 7\theta$.

4. Prove that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$, and find the value of—

$$\cot^{-1}\frac{ab+1}{a-b} + \cot^{-1}\frac{bc+1}{b-c} + \cot^{-1}\frac{ca+1}{c-a}.$$

5. Show that in any triangle—

$$a\sin A + b\sin B + c\sin C = \sqrt{a^2+b^2+c^2}\ \sqrt{\sin^2 A + \sin^2 B + \sin^2 C}.$$

If D, E, F are the feet of the perpendiculars from A, B, C on a, b, c, and R_a, R^b, R_c the radii of the circles described about the triangles AFE, BDF, CED respectively, then $R_a : R_b : R_c :: \cos A : \cos B : \cos C$.

6. A person standing due south of a lighthouse observes that his shadow cast by the light at the top is 24 feet long; on walking 100 yards due east he finds his shadow to be 30 feet. Supposing him to be 6 feet high, find the height of the light from the ground.

7. In a triangle ABC, if BC and the angles ABC and ACB are given, show how to find the remaining sides: thus $BC=1652$, angle $ABC = 26^\circ\ 30'$, and angle $ACB = 47^\circ\ 15'$; find AB and AC.

L sin 73° 45′ = 9·9822938 log 1652 = 3·2180100.
L sin 26° 30′ = 9·6495274 log 7678 = 3·8852481, $D=57$.
L sin 47° 15′ = 9·8658868. log 12636 = 4·1016096, $D=344$.

II.

*1. A, B, and C are the angles of a triangle; the number of grades in A and C is four times the number of those in B, and the number of degrees in B and C is double that in A. Find the angles in degrees, grades, and circular measure.

2. Define the cosecant of an angle, and trace the changes in its sign and magnitude as the angle increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

Prove geometrically that $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

3. Write down all the values of θ which satisfy the equations:—

(i.) $\operatorname{cosec}^2 \theta = \frac{2 \sec \theta}{\tan^2 \theta}$.

(ii.) $\sin^4 \theta - \cos^4 \theta = 1$.

(iii.) $\sin^2 2\theta - \sin^2 \theta = \frac{1}{4}$.

*4. If θ be less than $\frac{\pi}{2}$, prove that $\sin \theta < \theta$, and $> \theta - \frac{\theta^3}{4}$, and explain the use of the formulae:—

$$\sin A + \sin (72° + A) - \sin (72° - A) = \sin (36° + A) - \sin (36° - A).$$

5. In any triangle prove that—

*(i.) $\frac{a^2 + b^2 - ab \cos C}{a \sin A + b \sin B + c \sin C} = \frac{a}{2 \sin A}$.

*(ii.) $\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$.

If in a triangle $A = 60°$, $b = 9$, $c = 6$, find the other angles, having given $\log 2 = \cdot 30103$, $\log 3 = \cdot 47712$, $L \tan 19° 6' = 9\cdot 53943$, $L \tan 19° 7' = 9\cdot 53984$.

6. Find the radii of the inscribed and escribed circles of a triangle.

*Prove that the area of the triangle formed by joining the points of contact of the inscribed circle $= 2r^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

7. The shadow of a tower is observed to be half the known height of the tower, and some time after to be equal to the full height: how much will the sun have gone down in the interval?

$\log 2 = \cdot 3010300$, $L \tan 63° 26' = 10\cdot 3009994$, diff. for $1' = \cdot 0003159$.

6. Given $\left.\begin{array}{l} b=8{\cdot}4 \\ c=12 \end{array}\right\}$ inches, $B=37^\circ\ 36'$, find A.

$\log 7 = {\cdot}845098.$
$\mathrm{L}\sin 37^\circ\ 36' = 9{\cdot}7854332.$
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diff. $1' = 711.$

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(i.) $\cos\theta = 0.$
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(iii.) $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta \,.\, \tan 4\theta \,.\, \tan 7\theta.$

4. Prove that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$, and find the value of—

$$\cot^{-1}\frac{ab+1}{a-b} + \cot^{-1}\frac{bc+1}{b-c} + \cot^{-1}\frac{ca+1}{c-a}.$$

5. Show that in any triangle—

$$a\sin A + b\sin B + c\sin C = \sqrt{a^2+b^2+c^2}\ \sqrt{\sin^2 A + \sin^2 B + \sin^2 C}.$$

If D, E, F are the feet of the perpendiculars from A, B, C on a, b, c, and R_a, R^b, R_c the radii of the circles described about the triangles AFE, BDF, CED respectively, then $R_a : R_b : R_c :: \cos A : \cos B : \cos C$.

6. A person standing due south of a lighthouse observes that his shadow cast by the light at the top is 24 feet long; on walking 100 yards due east he finds his shadow to be 30 feet. Supposing him to be 6 feet high, find the height of the light from the ground.

7. In a triangle ABC, if BC and the angles ABC and ACB are given, show how to find the remaining sides: thus $BC = 1652$, angle $ABC = 26^\circ\ 30'$, and angle $ACB = 47^\circ\ 15'$; find AB and AC.

$\mathrm{L}\sin 73^\circ\ 45' = 9{\cdot}9822938$ $\quad\log 1652 = 3{\cdot}2180100.$
$\mathrm{L}\sin 26^\circ\ 30' = 9{\cdot}6495274$ $\quad\log 7678 = 3{\cdot}8852481,\quad D = 57.$
$\mathrm{L}\sin 47^\circ\ 15' = 9{\cdot}8658868.$ $\quad\log 12636 = 4{\cdot}1016096,\quad D = 344.$

II.

*1. A, B, and C are the angles of a triangle; the number of grades in A and C is four times the number of those in B, and the number of degrees in B and C is double that in A. Find the angles in degrees, grades, and circular measure.

2. Define the cosecant of an angle, and trace the changes in its sign and magnitude as the angle increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

Prove geometrically that $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

3. Write down all the values of θ which satisfy the equations:—

(i.) $\operatorname{cosec}^2\theta = \frac{2\sec\theta}{\tan^2\theta}$.

(ii.) $\sin^4\theta - \cos^4\theta = 1$.

(iii.) $\sin^2 2\theta - \sin^2\theta = \frac{1}{4}$.

*4. If θ be less than $\frac{\pi}{2}$, prove that $\sin\theta < \theta$, and $> \theta - \frac{\theta^3}{4}$, and explain the use of the formulae:—

$$\sin A + \sin(72^\circ + A) - \sin(72^\circ - A) = \sin(36^\circ + A) - \sin(36^\circ - A).$$

5. In any triangle prove that—

*(i.) $\frac{a^2 + b^2 - ab\cos C}{a\sin A + b\sin B + c\sin C} = \frac{a}{2\sin A}$.

*(ii.) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$.

If in a triangle $A = 60^\circ$, $b = 9$, $c = 6$, find the other angles, having given $\log 2 = \cdot30103$, $\log 3 = \cdot47712$, $\text{L}\tan 19^\circ 6' = 9\cdot53943$, $\text{L}\tan 19^\circ 7' = 9\cdot53984$.

6. Find the radii of the inscribed and escribed circles of a triangle.

*Prove that the area of the triangle formed by joining the points of contact of the inscribed circle $= 2r^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

7. The shadow of a tower is observed to be half the known height of the tower, and some time after to be equal to the full height: how much will the sun have gone down in the interval?

$\log 2 = \cdot3010300$, $\text{L}\tan 63^\circ 26' = 10\cdot3009994$, diff. for $1' = \cdot0003159$.

XIII.

1. The angle subtended at the centre of a circle by an arc which is equal in length to the radius is invariable.

Express in degrees and circular measure the vertical angle of an isosceles triangle which is half of each of the angles at the base.

* 2. Investigate an expression which shall include all angles that have a given cosine.

3. Prove the following formulæ—

(i.) $(\sin A + \cos A)(\tan A + \cot A) = \sec A + \operatorname{cosec} A$.
(ii.) $\sin 3A = 3 \sin A - 4 \sin^3 A$.
(iii.) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2} = \frac{\pi}{4}$.

4. Find the values of sin 18° and sin 48°.

5. Prove that $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$: and, if $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$, the sides are in arithmetical progression.

6. Two sides of a triangle are 1·5 and 13·5 respectively, and the included angle is 65°, find the remaining angles, having given

log 2 = ·3010300
L cot 32° 30′ = 10·1958127
L tan 51° 28′ = 10·0988763
L tan 51° 29′ = 10·0991355.

* 7. Find the area of a regular polygon inscribed in a circle, and prove that the square of the side of an inscribed pentagon equals the sum of the squares of the sides of a hexagon and decagon inscribed in the same circle.

* 8. Eliminate θ between the equations—

$$m = \operatorname{cosec}\theta - \sin\theta, \text{ and } n = \sec\theta - \cos\theta.$$

XIV.

1. Show that the circumference of a circle always bears a constant ratio to the diameter, and explain what is meant by the circular measure of an angle.

Express in circular measure an angle of 240°, and express in degrees the angle whose circular measure is $\frac{2\pi}{3}$.

* 2. Establish the formulæ :—

$$\text{(i.)}\ \sin(A+B)=\sin A\cos B+\cos A\sin B.$$

$$\text{(ii.)}\ \sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

3. Show that—

$$\text{(i.)}\ \tan(45^\circ+A)\tan(45^\circ-A)=1.$$

$$\text{(ii.)}\ \cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}.$$

$$\text{(iii.)}\ \sin 2A=2\,\frac{\cot A+\tan A}{\sec^2 A+\operatorname{cosec}^2 A}.$$

4. If $A+B+C=180^\circ$, show that $\tan A+\tan B+\tan C=\tan A\tan B\tan C$, and that $\dfrac{\sin A+\sin B+\sin C}{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$.

* 5. Find an expression for all the angles which have a given tangent. Solve the equation $\operatorname{cosec}\theta=1+\cot\theta$.

6. One of the sides of a right-angled triangle is $\frac{2}{7}$ of the hypothenuse ; find the other angles, having given

$\log 2=\cdot301030$ $\quad$ $L\sin 14^\circ\,11'=9\cdot455921$
$\log 7=\cdot845098$ $\quad$ $L\sin 14^\circ\,12'=9\cdot456031.$

7. Two sides of a triangle are 4 feet and 6 feet in length respectively, and the included angle is 30° ; find the area of the triangle.

* 8. Show that the length of a side of an equilateral triangle inscribed in a circle is to that of a side of a square inscribed in the same circle as $\sqrt{3}$ to $\sqrt{2}$.

XV.

1. Describe the three different systems of units by means of which the magnitudes of angles are expressed.

* Express in each of the three systems the angle between two adjacent sides of a regular hexagon.

2. Trace the changes in sign and magnitude of $\cot A$ as A changes from 90° to 270°.

3. Prove that—

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\ \tan B},$$

and that

$$\tan 15°=\frac{\tan 60°+\tan 30°-2}{\tan 60°-\tan 30°}.$$

4. Prove that—

(i.) $\sec^2 A-\sec^2 B=\tan^2 A-\tan^2 B$.

(ii.) $\sin(A+B)\sin(A-B)=\cos^2 B-\cos^2 A$.

(iii.) $\tan^{-1}\dfrac{1}{1+a}+\tan^{-1}\dfrac{1}{1-a}+\tan^{-1}\dfrac{2}{a^2}=n\pi$.

5. Solve the equation $\cot\theta-\tan\theta=2$.

6. Two sides of a triangle are as 5 : 9, and the included angle is a right angle. Find the other angles, having given

$\log 2=\cdot 3010300$	$L\tan 19°\ 26'=9\cdot 7446051$
$\log 3=\cdot 4771213$	$L\tan 19°\ 27'=9\cdot 7448497$.

7. Find an expression for the radius of the circle described about a given triangle.

If the altitude of an isosceles triangle is equal to its base, the radius of the circumscribing circle is $\frac{5}{8}$ of the base.

8. Prove that the area of the square described about a circle is double that of the square inscribed in the same circle.

XVI.

1. The angle subtended at the centre of a circle by an arc, which is equal in length to the radius, is invariable.

Express seven-sixteenths of a right angle in circular measure and in grades.

2. Trace the variations in sign and magnitude of $\cos A - \sin A$, as A varies from 0° to 180°.

3. Express the sine and cosine of the difference of two angles in terms of the sines and cosines of the angles.

Find sin 75°.

4. Prove the following formulæ—

(i.) $\operatorname{vers}(A+B)\operatorname{vers}(A-B) = (\cos A - \cos B)^2$.

(ii.) $\sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) = \frac{\sec 2\theta}{2}$.

* (iii.) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$.

* 5. Find the general value of A, when $\cos 2A = \sin A$.

* 6. Solve the triangle in which $a=18$, $b=12$, $c=10$, having given—

tan 54° 44′ = 1·4140943 tan 15° 47′ = ·2826573
tan 54° 45′ = 1·4149673 tan 15° 48′ = ·2829715.

* 7. Find the radii of the inscribed and escribed circles of a triangle, and if they be called r, r_1, r_2, r_3, prove that—

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

* 8. An equilateral triangle and a regular hexagon have the same perimeter; show that the areas of their inscribed circles are as 4 : 9.

XVII.

1. What are the units of angular measure commonly employed?

Find the circular measure of $60°$, $\pi°$, and the number of degrees in the angle whose circular measure is ·7854.

* 2. Define the trigonometrical functions of an angle, and find those of the angle $60°$.

3. Prove the formulæ—

$$\text{(i.)}\ \cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B,$$

$$\text{(ii.)}\ \frac{\cos 2A-\cot 2A}{\operatorname{cosec} 2A+\cot 2A}=\tan^2 A\ ;$$

and find x and y from the equations :—

$$\frac{x}{a}\cos\frac{\phi+\phi_1}{2}+\frac{y}{b}\sin\frac{\phi+\phi_1}{2}=\cos\frac{\phi-\phi_1}{2},\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

4. Explain the ambiguous case in the solution of triangles.

Having given $A=30°$, $a=\sqrt{2}$, $c=2$, solve the triangle.

5. If θ be the circular measure of an angle less than a right angle, then $\sin\theta<\theta$, and $>\theta-\frac{\theta^3}{4}$.

6. Find the radius of a circle inscribed in a given triangle.

If the radius of the inscribed circle is equal to half that of the circumscribed circle, the triangle is equilateral.

* 7. Define $\tan^{-1}x$, and prove that—

$$\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}.$$

XVIII.

1. If we take for the unit angle the angle between two consecutive sides of a regular hexagon, how many units of this kind are there in (1.) a right angle; (2.) the unit of circular measure?

2. Find the sine and cosine of any angle in terms of its cotangent.

If $2 \sec\theta = \tan\alpha + \cot\alpha$; find $\tan\theta$ and $\operatorname{cosec}\theta$.

3. Prove that—

$$\sin(90^\circ + A) = -\cos(180^\circ + A), \text{ and } \cos(90^\circ + A) = \sin(180^\circ + A).$$

4. Find the sine and cosine of 30°, 45°, and 15°.

* 5. Prove geometrically that—

$$\sin 2A = 2\sin A\cos A, \text{ and } \cos 2A = \cos^2 A - \sin^2 A.$$

If the first of these equations be solved for $\sin A$ so as to express it in terms of $\sin 2A$, there are four values: show from other considerations that this ought to be the case, and find the four values.

6. Prove that (i.) $\tan^2\dfrac{A}{2} = \dfrac{\sec A - 1}{\sec A + 1}$.

* (ii.) $(2\cos A + 1)(2\cos A - 1)(2\cos 2A - 1) = 2\cos 4A + 1$.

7. Express the tangent of the half angle of a triangle in terms of the sides. The sides being 20, 21, and 29, find the angle subtended by 29. Find also the area of this triangle.

8. Two sides of a triangle are 9 and 7 feet respectively, and the angle between them is 60°; find the other angles, having given

$\log 2 = \cdot 3010300$ $\quad$ $\text{L} \tan 12^\circ\, 12' = 9\cdot 3348711$
$\text{L} \cot 30^\circ = 10\cdot 2385606$ $\quad$ $\text{L} \tan 12^\circ\, 13' = 9\cdot 3354823$.

XIX.

1. Define the tangent of an angle; find the value of $\tan 60^\circ$, and prove that $\tan(180^\circ - A) = -\tan A$, and $\tan(90^\circ + A) = -\cot A$. Also trace the changes in its sign and magnitude, as A varies from 0° to 180°.

2. If A and B be each less than a right angle, and $A > B$, prove that $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

Draw the figure for a case (i.) where A is $>$ a right angle; (ii.) where A is $< B$.

* 3. If the sine of an angle is given, show how to find the sine of half the angle, and explain why the general expression obtained ought to include four values.

Given $\sin 300^\circ = \frac{-\sqrt{3}}{2}$, find $\sin 150^\circ$.

4. Prove (i.) $\sec A = \dfrac{1 + \tan^2 \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$,

$$\text{(ii.) } \operatorname{cosec} A = \frac{1}{2}\left(\tan\frac{A}{2} + \cot\frac{A}{2}\right),$$

and solve the equation $\sin x \cos x + a \sin^2 x = b$.

5. The angles in one regular polygon are twice as many as in another regular polygon; and an angle of the former is to an angle of the latter as 3 : 2. Find the number of sides.

6. In an oblique-angled triangle, given $a = 145$, $b = 178$, $B = 41^\circ\ 10'$, find A.

$\log 178 = 2{\cdot}2511513$ $\quad$ $L \sin 41^\circ\ 10' = 9{\cdot}8183919$

$\log 145 = 2{\cdot}1613680$ $\quad$ $L \sin 32^\circ\ 21'\ 54'' = 9{\cdot}7286086$

and show whether there is any ambiguity in the result.

* 7. In any triangle, given a, b, and A, find an equation for determining c, and discuss its values, according as a is equal to, less than, or greater than $b \sin A$.

XX.

1. Trace the changes of sign of $\sin\theta$, as θ increases through four right angles.

2. Find the values of $\sin 30^\circ$ and $\sin 18^\circ$, and prove that a value of θ which satisfies the equation $2\tan\theta=\cos\theta$ lies between 18° and 30°.

3. If $\cos\theta=\dfrac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$, then $\tan\dfrac{\theta}{2}=\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}$.

Prove also that if $A+B+C=180^\circ$,

$$\frac{\cos A+\cos B+\cos C-1}{\sin A+\sin B+\sin C}=\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2},$$

and that $\tan^{-1}x+\tan^{-1}y+\tan^{-1}\dfrac{1-x-y-xy}{1+x+y-xy}=\dfrac{\pi}{4}$.

4. In a plane triangle the two sides are $\sqrt{3}+1$ and $\sqrt{3}-1$, and the included angle is 60°; find the remaining angles.

* 5. If a, b, c be the sides of a triangle, and α, β, γ the perpendiculars drawn upon them from the centre of the circumscribing circle, prove that $4\left\{\dfrac{a}{\alpha}+\dfrac{b}{\beta}+\dfrac{c}{\gamma}\right\}=\dfrac{abc}{\alpha\beta\gamma}$.

6. Find the area of a triangle in terms of its perimeter and the radius of the inscribed circle.

The area of an equilateral triangle described about a circle is four times the area of an equilateral triangle inscribed in the same circle.

7. A person wanting to calculate the perpendicular height of a cliff, takes its angular altitude 12° 30′, and then measures 950 yards in a direct line towards the base, when he is stopped by a river; he then takes a second altitude, and finds it 69° 30′. Find the height of the cliff.

Given $\log 5=\cdot6989700$ $\quad$ $L\sin 12^\circ\,30'=9\cdot3353368$
$\log 19=1\cdot2787536$ $\quad$ $L\cos 33^\circ=9\cdot9235914$
$\log 2296=3\cdot3610566$ $\quad$ $L\cos 20^\circ\,30'=9\cdot9715876$.

XXI.

1. Show that there are in general two different angles between 0° and 360° which have a given sine, or a given cosine; but that there is only one angle between 0° and 360°, which has both a given sine, and a given cosine.

Given the tangent of an angle, express in terms of it the other trigonometrical ratios of the angle.

2. Establish the formulæ—

(i.) $\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2}$.

(ii.) $\cos 3A = 4 \cos^3 A - 3 \cos A$;

and employ (ii.) to obtain the values of cos 18° and sin 18°.

3. Prove the equalities—

(i.) $\frac{\sin 2A}{1+\cos 2A} \cdot \frac{\cos A}{1+\cos 2A} = \tan \frac{A}{2}$.

* (ii.) $\sin(B-C) + \sin(C-A) + \sin(A-B) + 4 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2} = 0$,

and solve the equation—

$$\sin(A+30°) + \cos(A+30°) = \sin(A-30°) + \cos(A-30°).$$

4. Prove that in any triangle—

(i.) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} =$ diameter of circumscribing circle.

* (ii.) $(b+c) \sin \frac{A}{2} = a \cos \frac{B-C}{2}$.

5. Given the three sides of a triangle; find the logarithmic sines of its half angles.

The sides of a triangle are 525 feet, 650 feet, and 777 feet respectively; determine its three angles.

6. *A*, *B*, *C* are three points in a straight line on a level piece of ground. A vertical pole is erected at *C*; the angle of elevation of its top, as observed from *A*, is 5° 30′; and, as observed from *B*, is 10° 45′; the distance from *A* to *B* being 100 yards. Find the distance *BC*, and the height of the pole.

7. Prove that the area of a circle is equal to the product of its semicircumference by its radius.

Three circles, each of radius 1 foot, touch one another externally; find the area of the curvilinear figure included between them.

[Logarithm Tables required.]

XXII.

1. Define the sine, cosine, and tangent of an angle; and trace the variation of the tangent, in sign and magnitude, through the four quadrants.

Given $\cos A = -\frac{5}{13}$; find $\sin A$ and $\cot A$; A being supposed less than 180°.

2. Prove that $\cos(A-B) = \cos A \cos B + \sin A \sin B$, and employ this formula to find the value of cos 15°.

3. Prove the following equalities—

(i.) $\sin(60^\circ + A) - \sin(60^\circ - A) = \sin A$.

(ii.) $(\cos A + \sin A \sin 2A)^2 - (\sin A + \cos A \sin 2A)^2 = (\cos 2A)^3$.

(iii.) $\tan(B-C) + \tan(C-A) + \tan(A-B) = \tan(B-C)\tan(C-A)\tan(A-B)$.

4. Show that in any triangle—

$$\text{(i.)}\ \cos A = \frac{b^2+c^2-a^2}{2bc}.$$

$$\text{(ii.)}\ \text{area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is half the sum of the sides; and find an expression for the radius of the circle which touches one side of the triangle and the other two produced.

5. Show how to solve a triangle when two sides and the included angle are given.

Given in a triangle ABC, $AB = 4517$ yards, $AC = 150$ yards, $A = 31^\circ\,30'$, find the remaining parts.

6. A, B are two points, 200 yards apart, on the bank of a river, and C is a point on the opposite bank. The angles ABC, BAC are respectively $54^\circ\,30'$ and $65^\circ\,30'$; find the breadth of the river.

* 7. Find an expression for the area of a regular polygon of n sides inscribed in a circle; and show that the square of the side of the inscribed pentagon is equal to the sum of the squares of the sides of the inscribed hexagon and decagon.

[Logarithm Tables required.]

XXIII.

1. Explain how the foreign or French method of measuring angles differs from the English method.

If an angle is given in one measure, investigate a rule for finding its value adapted to the other measure. Express in foreign measure each angle of the isosceles triangle described in Euclid's 10th Proposition, Book IV.

* 2. Show how to express the cosine, tangent, and secant of an angle in terms of the sine. The sine of an angle less than 90° is $\frac{3}{5}$, find the cosine, tangent, and secant.

3. Investigate the formula for $\sin(A-B)$ in terms of the sines and cosines of A and B; express the formula also in terms of the cosines of A and B only.

Prove $\cos 15^\circ \times \cos 75^\circ = \frac{1}{4}$.

4. Prove $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$. If A be the angle of an equilateral triangle, find $\tan A$ from the above formula.

5. If A, B, C be the angles of a plane triangle subtended by the sides a, b, c respectively, prove $\frac{\sin B}{\sin C} = \frac{b}{c}$, and thence deduce the formula $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$: show how this is applicable to the solution of a triangle of which the data are two sides and the included angle.

6. The base of an isosceles triangle is 100 feet, and the vertical angle is 125°; solve the triangle.

7. In a triangle ABC, given $AC = 166{\cdot}5$ feet, $BC = 162{\cdot}5$ feet, the angle $A = 52^\circ\ 19'$, solve *either* of the triangles to which the data belong.

[Logarithm Tables required.]

XXIV.

1. Define circular measure, and show how to convert degrees into circular measure.

In the pentagon $ABCDE$ all the angles are equal, and AF and BF are drawn perpendicular to EA and BC, meeting in the point F. Find the circular measure of the angle AFB.

* 2. Express each of the trigonometrical functions in terms of the tangent.

If $\operatorname{cosec} A = \frac{2}{\sqrt{3}}$, find $\sec A$, supposing that A lies between 90° and 180°.

3. Find $\tan(A+B)$ in terms of the tangents of A and B.

Prove that $\tan 15° + \tan 75° = 4$.

4. Find $\sin 3A$ and $\cos 3A$.

* Prove that $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2$.

5. In any triangle ABC, whose sides are a, b, and c, prove that

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

6. Assuming the result of the last question, find a formula for $\cos \frac{A}{2}$ in terms of the sides adapted to logarithmic computation.

7. In any triangle, if $A = 47°$, $a = 180$, $b = 215$, find B.

Given that $L \sin 47° = 9{\cdot}864127$ $\qquad \log 180 = 2{\cdot}255273$
$L \sin 60°\,52' = 9{\cdot}941258$ $\qquad \log 215 = 2{\cdot}332438.$
$L \sin 60°\,53' = 9{\cdot}941328$

* 8. If $A = 55°$, $B = 65°$, and $C = 270$, find a.

Given that $L \sin 55° = 9{\cdot}9133645$
$\log 3 = {\cdot}4771213$ $\qquad \log 2{\cdot}5553 = {\cdot}4070508$
$\log 2 = {\cdot}3010300$ $\qquad \log 2{\cdot}5554 = {\cdot}4072209.$

* 9. The sides a, b, and c of the triagle ABC are to each other as $4:5:6$. Find B.

Given that $L \cos 27°\,53' = 9{\cdot}9464040$
$L \cos 27°\,54' = 9{\cdot}9463371$
$\log 2 = {\cdot}3010300.$

XXV.

1. Is any unit of angular measurement referred to in the First Book of Euclid? What units of angular measurement are commonly used in English treatises on Plane Trigonometry? Examine the relation of these units to each other.

Find the circular measure of an angle of an equilateral and equiangular pentagon.

2. Define the tangent and cotangent of an angle. Express the values of $\cot 90°$, $\tan 180°$, $\tan 270°$. Show that the tangent of any angle will have the same sign as its cotangent.

3. Prove by means of a geometrical construction—

$$\cos(A+B)=\cos A\cos B-\sin A\sin B,$$

and explain the result if $B=A$. Find $\cos 15°$.

4. Find $\tan 3A$ in terms of $\tan A$, and from the formula obtained determine the numerical value of $\tan A$, if $3A=90°$.

5. Prove $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\dfrac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}$. Also if a and b are the sides of a triangle subtending the angles A and B respectively, prove either of the above expressions equal to $\dfrac{a+b}{a-b}$.

6. In any triangle prove (i.) $a\sin B-b\sin A=0$.
(ii.) $a\cos B+b\cos A=c$.

* Hence find $\cos A$ in terms of the sides a, b, c.

7. The side of an equilateral triangle is 20 feet, find the numerical value of the radius of the circle circumscribing the triangle.

8. Find the least angle of the triangle whose three sides are 200, 250, 300 feet respectively.

9. Two straight railroads are inclined at an angle of 20° 16′. At the same instant, from their point of intersection, an engine starts along each line; one travels at the rate of 20 miles an hour; at what rate must the other travel, so that after three hours the engines shall be at a distance from each other of 30 miles? Show that there are two solutions.

[Logarithm Tables required.]

XXVI.

1. Define the circular measure of an angle.

* If the number of degrees in an angle be equal to the number of grades in the complement of the same angle, prove that the circular measure of the angle is $\frac{5\pi}{19}$.

* 2. Define the principal trigonometrical ratios, and express each of them in terms of the tangent.

Find sin 60° and cot 30°.

3. Find $\sin(A+B)$ in terms of the sines and cosines of A and B, stating the limitations in the values of A and B assumed in the proof. Apply to find sin 75°.

4. Find $\tan 2A$ and $\sin 3A$.

* 5. Given $\tan\frac{A}{2}=\frac{a}{b}$, find $\sin A$ and $\sin 2A$.

6. In any triangle, of which the semi-perimeter is s, prove the formula $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$.

7. From B and A the acute angles of the right-angled triangle ABC, the lines BD and AE are drawn to the opposite sides of the triangle, such that the angles DBC and EAC are each equal to α; prove that $DE=AB\tan\alpha$.

* 8. Given $A=40°$, $a=140{\cdot}5$, $b=170{\cdot}6$. Find B and C.

Given that $L\sin 40°=9{\cdot}8080675$ $\quad\log 1405=3{\cdot}1476763$

$L\sin 51°\,18'=9{\cdot}8923342$ $\quad\log 1706=3{\cdot}2319790$.

$L\sin 51°\,19'=9{\cdot}8924354$

9. Two sides of a triangle are 85 and 75 yards respectively, and the included angle is 70°; find the remaining angles.

Given that $\log 160=2{\cdot}20412$ $\quad L\tan 5°\,6'=8{\cdot}9505967$

$L\cot 35°=10{\cdot}1547732$ $\quad L\tan 5°\,7'=8{\cdot}9520211$.

XXVII.

1. Define the cosine and versed sine of an angle, and trace the changes in magnitude and sign of the latter, as the angle increases from 0° to 360°.

2. Prove that $\sin\left(\frac{\pi}{2}+A\right)=\cos A$, and that $\cos\left(\frac{\pi}{2}+A\right)=-\sin A$; and find an expression for all the angles which have a given tangent.

* 3. Investigate a formula for expressing the cosine of half an angle in terms of the sine of the whole angle; and if the angle lies between 270° and 360°, show which signs of the roots must be taken.

4. Explain clearly the ambiguous case in the solution of triangles.

If $AB=204$, $AC=145$, and the angle $B=35°$, find BC, and the angles A and C.

Given		
	$\log 204=2{\cdot}30963$	$\text{L} \sin 35°=9{\cdot}75859$
	$\log 145=2{\cdot}16136$	$\text{L} \cos 36° 12'=9{\cdot}90686$
	$\text{L} \cos 1° 12'=9{\cdot}99990$	$\text{L} \sin 18° 48'=9{\cdot}50821$
	$\log 252{\cdot}744=2{\cdot}40267$	$\log 81{\cdot}4687=1{\cdot}91098.$

5. Compare the areas of the circles inscribed in, and described about, a given triangle.

* 6. Find all the values of θ which satisfy the equation—

$$\cos\theta+\sin\theta=\sqrt{2}.$$

7. Prove that—

$$\cos^{-1}\frac{1-m^2}{1+m^2}-\cos^{-1}\frac{1-n^2}{1+n^2}=\tan^{-1}\frac{2(m-n)(1+mn)}{(1+mn)^2-(m-n)^2}.$$

And that—

$$\frac{\text{vers } A}{\text{vers } B}=\frac{a(a+c-b)}{b(b+c-a)}$$

XXVIII.

1. Define the grade and the degree, and express the interior angle of a regular pentagon both in grades and degrees.

2. Define the circular measure of an angle, and find two angles such that their difference is one degree, and their sum is the unit of circular measure.

* 3. Define the principal trigonometrical ratios, and find the sine of 30°, and secant of 45°.

* 4. Prove that $\sin(A-B) = \sin A \cos B - \cos A \sin B$, stating what limitations your proof supposes in the values of A and B respectively.

5. Find $\sin 2A$ and $\cos 3A$.

6. If $\sin A = \frac{2ab}{a^2+b^2}$, find $\tan\frac{A}{2}$.

7. Prove that the area of any triangle, whose sides are a, b, and c, is $\sqrt{s(s-a)(s-b)(s-c)}$, s being the semi-perimeter.

* 8. In the triangle ABC, $A=60°\ 15'$, $B=54°\ 30'$, and $AB=100$ yards; find AC, having given—

$\text{L} \sin 54°\ 30' = 9\cdot9106860$ $\log \cdot89646 = \bar{1}\cdot9525317$.
$\text{L} \sin 65°\ 15' = 9\cdot9581543$

9. The sides and the included angle of a triangle are 327 feet, 256 feet, and 56° 28′ respectively, find the remaining angles.

Given $\log \cdot71 = \bar{1}\cdot8512583$ $\text{L} \tan 61°\ 46' = 10\cdot2700705$
$\log \cdot583 = \bar{1}\cdot7656686$ $\text{L} \tan 12°\ 46' = 9\cdot3552267$
$\text{L} \tan 12°\ 47' = 9\cdot3558126$.

10. ABC is a triangle, and D the middle point of the base BC, prove that the sine of BAD is—

$$\frac{b \sin A}{\sqrt{b^2+2bc\cos A+c^2}}.$$

XXIX.

1. Define a degree, a grade, and the unit of circular measure.

The number that denotes an angle when expressed in circular measure is less by 1·5 than that denoting the number of degrees in the same angle; find the angle.

2. Express all the trigonometrical ratios in terms of the sine.

If the cosecant be twice the sine, find the angle.

3. Prove that—

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

stating the limitations assumed in the proof as to the values of A and B.

4. Find $\cos 2A$ and $\sin 3A$.

5. If any arc of a circle be divided into two equal and also into two unequal parts, prove that the rectangle of the chords of the two unequal parts, together with the square of the chord of the arc between the points of section, is equal to the square on the chord of half the arc.

6. Prove that in any triangle—

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

7. In the triangle ABC, $A = 26^\circ\ 26'$, $b = 127$, and $a = 85$; find B.

Given $\log 1·27 = ·1038037$ $\quad$ $L \sin 26^\circ\ 26' = 9·6485124$

$\log 8·5 = ·9294189$ $\quad$ $L \sin 41^\circ\ 41' = 9·8228302$

$L \sin 41^\circ\ 42' = 9·8229721.$

8. If one angle of a triangle be 65°, and one of the sides containing this angle be four times the other, find the remaining angles.

Given $\log 2 = ·3010300$ $\quad$ $L \cot 32^\circ\ 30' = 10·1958127$

$\log 3 = ·4771213$ $\quad$ $L \tan 43^\circ\ 17' = 9·9739602$

$L \tan 43^\circ\ 18' = 9·9742133.$

* 9. The angle of elevation of a tower at a distance of 20 yards from its foot is three times as great as the angle of elevation 100 yards from the same point. Show that the height of the tower is $\frac{300}{\sqrt{7}}$ feet.

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(*Obligatory.*)

XXX.

1. Express in circular measure and in degrees to three places of decimals an angle which is subtended at the centre of a circle 3 square inches in area by an arc of 1 inch. $\left[\pi=\frac{22}{7}\cdot\right]$

2. Define the trigonometrical ratios of A involved in the equation-

$$\cot A+\tan A=\sec A \operatorname{cosec} A,$$

and establish its truth by a geometrical construction.

3. Show how the formula for tan $(A+B)$ in terms of tan A, tan B, may be deduced from the formula for sin $(A+B)$.

Evaluate $\tan\left(\frac{\pi}{4}+A\right)\times\tan\left(\frac{3\pi}{4}+A\right)\cdot$

4. Prove that $\sin 18^\circ=\frac{\sqrt{5}-1}{4}\cdot$

Hence find an expression for sin 3°.

5. Prove that—

(i.) $\tan^2 60^\circ-2\tan^2 45^\circ=\cot^2 30^\circ-2\sin^2 30^\circ-\frac{3}{4}\operatorname{cosec}^2 45^\circ.$

(ii.) $\sec 2A-\frac{1}{2}\tan 2A\sin 2A=\frac{\cot^2 A+\tan^2 A}{\cot^2 A-\tan^2 A}\cdot$

(iii.) $\tan^{-1}\frac{1}{2}+\cot^{-1}\frac{1}{3}+\sin^{-1}\frac{7\sqrt{2}}{10}=0.$

6. Find all the values of A which satisfy the equation—

$$\cos^3 A - \cos A \sin A - \sin^3 A = 1.$$

7. If x be the least positive angle of which the sine is equal to $\sin A$, find a general formula for A in terms of x.

Find also all possible values of $\sin \frac{A}{6}$ in terms of the trigonometrical ratios of $\frac{x}{6}$.

8. Prove that in any triangle ABC, with the usual notation—

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

If AD be drawn perpendicular to the plane of ABC, and DB, DC be joined, show that—

$$\sin DCB \,.\, \sin DBA = \sin DCA \,.\, \sin DBC.$$

9. If the sides of a triangle be 7·152 inches, 8·263 inches, 9·375 inches, find its area.

$\log 1{\cdot}2395 = {\cdot}0932465$
$\log 5{\cdot}243 = {\cdot}7195799$
$\log 4{\cdot}132 = {\cdot}6161603$
$\log 3{\cdot}02 = {\cdot}4800069$
$\log 2{\cdot}8477 = {\cdot}4544942$, diff. = 152.

10. Find the radii of the three circles escribed to a triangle in terms of the sides.

Show that the sum of the products of these radii two and two is equal to the square on the semi-perimeter of the triangle.

11. A straight flagstaff, leaning due east, is found to subtend an angle a at a point, in the plane upon which it stands, a yards west of the place. At a point b yards east of the base the flagstaff subtends an angle β. Find at what angle it leans.

XXX.*

1. Assuming 3·1416 as the value of π, express in circular measure 3° 12′, and find to seconds the angle whose circular measure is ·8.

Find approximately the distance of a tower whose height is 51 feet, and which subtends at the eye an angle of $5\frac{5}{11}'$.

2. Prove that $\cos(135° + A) + \sin(135° - A) = 0$.

If $\cos A = \tan B$, and $\sin B = \cot A$, find A and B.

3. Find the smallest possible value of θ which satisfies the equation $\cos\theta = \sin\left((4n+3)\dfrac{\pi}{2} + a\right)$.

Trace the changes in the sign and value of $\cos(\pi\sin\theta)$ as θ increases from 0° to 180°.

4. Prove that $\cos 2A = \dfrac{1 - 6\tan^2\dfrac{A}{2} + \tan^4\dfrac{A}{2}}{\left(1 + \tan^2\dfrac{A}{2}\right)^2}$, and that the value of $\sin(n+1)B\sin(n-1)B + \cos(n+1)B\cos(n-1)B$ is independent of n.

If $\dfrac{\tan(\theta - 15°)}{\tan(\theta + 15°)} = \dfrac{1}{3}$, find θ.

5. From the extremities of a diameter of a circle AB, chords AD, BE are drawn intersecting in C at an angle of 120°; if the area of ABD is double that of AED, find the angles at A and B.

6. If $\sin A = a$, prove that—

$$2\cos\frac{A}{2} = \pm\sqrt{1+a} \pm \sqrt{1-a},$$

and find in what quadrant A is when the two negative signs have to be taken.

If $\cos A = \tan A$, prove that $\sin A = 2\sin 18°$.

7. Find $\log_{10} \cdot 0001$ and $\log_{3\sqrt{11}} \{297\sqrt{11}\}^{\frac{2}{3}}$.

Find also $\log \frac{1}{147}$ and $\sqrt{1 \cdot 323}$,

$$\begin{aligned} \text{having given } \log 3 &= \cdot 4771213 \\ \log 7 &= \cdot 845098 \\ \log 3 \cdot 6593 &= \cdot 563399. \end{aligned}$$

8. Prove that in any plane triangle—

(i.) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

$\left(\text{assuming } \cos A = \frac{b^2+c^2-a^2}{2bc}\right)$.

(ii.) Area of triangle $= \frac{a^2}{2} \cdot \frac{\sin B \cdot \sin C}{\sin (B+C)}$.

9. The angles A, B, C of a triangle ABC are 40°, 60°, 80° respectively, and CD is drawn from C to the base bisecting the angle ACB; find CD, having given—

$AB = 100$ inches $\qquad$ $\text{L} \sin 40° = 9 \cdot 8080675$

$\log 2 = \cdot 30103$ $\qquad$ $\text{L} \sin 50° = 9 \cdot 884254$

$\log 5 \cdot 73979 = \cdot 7588951$ $\qquad$ $\text{L} \sin 60° = 9 \cdot 9375306$.

10. Two lines of straight railway, ABC, DEC, meet at C, telegraph posts being situated at A, B, D, E; the angles DAE, DBE are each equal to α; and the angles EAB, EBC are β and γ respectively; show that—

$$BC = AB \frac{\sin \beta \sin (\alpha+\beta)}{\sin (\gamma-\beta) \sin (\alpha+\beta+\gamma)}.$$

11. Find the radius of a circle inscribed in a right-angled triangle in terms of the sides.

Two equal right-angled triangles ABC, $A'B'C$ have a common right angle C, and their hypothenuses meet in D, CA, CA' being equal; compare the radii of the circles inscribed in ABC and $A'BD$.

XXXI.

1. Define the "unit of circular measure," and find how many degrees, minutes, and seconds it contains, neglecting fractions of a second, and assuming the ratio of the circumference to the diameter of a circle to be 22 : 7. Also find the circular measure of 49° 41′ 15″.

2. Prove that $\cos A = -\cos(180^\circ - A)$.

If the sum of four angles is 180°, show that the sum of the products of their sines, taken two together, is equal to the sum of the products of their cosines, taken two together.

3. Find an expression for all angles that have a given tangent.

Find all the values of θ which satisfy $\operatorname{cosec}\theta = \cot\theta + \sqrt{3}$.

4. Prove that— (i.) $\cot^2(45^\circ + A) = \sec^2(45^\circ - A) - 1$.

(ii.) $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$.

(iii.) $\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{2}{9} = \dfrac{1}{2}\cos^{-1}\dfrac{3}{5}$.

5. Find the acute angles of a right-angled triangle whose hypothenuse is four times as long as the perpendicular let fall upon it from the opposite angle.

6. Find the values of $\sin 18^\circ$ and $\cos 36^\circ$.

Prove that $\cos 12^\circ + \cos 60^\circ + \cos 84^\circ = \cos 24^\circ + \cos 48^\circ$.

7. Show how the characteristic of any logarithm to the base 10 may be found by inspection.

Given $\log 5 = {\cdot}69897$, find $\log 200$, $\log {\cdot}025$, $\log \sqrt[5]{62{\cdot}5}$, and also $\mathrm{L}\sin 30^\circ$ and $\mathrm{L}\cos 45^\circ$.

8. Show that in any plane triangle—

(i.) $2bc\cos A = b^2 + c^2 - a^2$

* (ii.) $(a+b+c)\left(\tan\dfrac{A}{2} + \tan\dfrac{B}{2}\right) = 2c\cot\dfrac{C}{2}$.

9. Given $a = 2b$, $C = 120^\circ$, $\log 3 = {\cdot}4771213$, $\mathrm{L}\tan 10^\circ\ 53' = 9{\cdot}283907$, diff. for $1' = 6808$; find A, B, and the ratio $c : a$.

10. At each end of a base of length $2a$, it is found that the angular altitude of a certain peak is θ, and at the middle point of the base the altitude is ϕ. Prove that the vertical height of the peak above the plane is $\dfrac{a\sin\theta\sin\phi}{\sqrt{\sin(\theta+\phi)\sin(\phi-\theta)}}$.

11. Express the radius of the inscribed circle of a triangle in terms of one side, and the two angles adjacent to it.

O is the centre of the inscribed circle of the triangle ABC. If R_1, R_2, R_3 be the radii of the circles circumscribing the triangles OBC, OCA, OAB respectively, and r, R the radii of the inscribed and circumscribed circles of the triangle ABC; show that $R_1R_2R_3=4rR^2$.

XXXII.

1. Assuming that a circle may be treated as a regular polygon with an infinite number of sides, show that the ratio of the circumference of a circle to its diameter is constant. What is the circular measure of the least angle whose sine is $\frac{1}{2}$, and what is the measure in degrees, etc., of the angle whose circular measure is $\frac{3\cdot1416}{20}$?

2. Show that the sine and the cosecant of any angle are respectively the same as the sine and cosecant of the supplement of the angle. Is this true of the cosine and secant of the angle?

Find the least positive value of A that satisfies the equation $2\sqrt{3}\cos^2 A=\sin A$, and *write down the general value of A if $\sec^2 A=\frac{4}{3}$.

* 3. Prove by a geometrical construction $\cos 2A=\cos^2 A-\sin^2 A$, without assuming or proving the ordinary formula for $\cos(A+B)$.

Prove $\cos 2A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$.

4. Prove—

$$\tan A=\frac{2\tan\frac{A}{2}}{1-\left(\tan\frac{A}{2}\right)^2}.$$

From this formula find $\tan\frac{A}{2}$ if $\tan A=\sqrt{3}$, and explain the double answer. Prove $\tan 75^\circ+\cot 75^\circ=4$.

5. Prove—

* (i.) $2\cos\frac{A}{2}=-\sqrt{1+\sin A}-\sqrt{1-\sin A}$ $\left\{\begin{array}{l}\text{if } A \text{ is between}\\ 270^\circ \text{ and } 360^\circ.\end{array}\right.$

(ii.) $\sin^{-1}\frac{2}{2\cdot5}+\sin^{-1}\frac{1\cdot5}{2\cdot5}=90^\circ$.

* (iii.) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$, if $A+B+C=180^\circ$.

6. What is meant by a system of logarithms?

Show that in all systems log 1 = 0, and that the logarithms of all proper fractions are negative.

Given $\log_{10} 2 = \cdot 3010300$, find $\log_{10} 125$, and express with its proper characteristic $\log_{10} \cdot 0005$; find the tabulated logarithm of sin 30°, given $L \sin 60° = 9\cdot 9375306$, find $\log_{10} 3$.

7. If A, B, C be the angles of a triangle, and a, b, c the sides opposite to them respectively, prove

$$\text{(i.)}\ \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

$$\text{(ii.)}\ b-c=a\cos\theta\,;\ \text{where}\ \tan\theta=\frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2}.$$

8. Find the radius of the circle circumscribing a triangle, in terms of its sides a, b, c.

If $c^2=a^2+b^2$, show that the expression obtained for the radius will become $\frac{c}{2}$.

9. For what data will the solution of a triangle become ambiguous? Explain this.

Given $B=30°$, $c=150$, $b=50\sqrt{3}$, show that of the two triangles that satisfy the data one will be isosceles and the other right-angled. Find the third side in the greatest of these triangles.

Would the solution be ambiguous if the data had been $B=30°$, $c=150$, $b=75$?

10. A flagstaff 100 feet high stands in the centre of an equilateral triangle which is horizontal; from the top of the flagstaff each side of the triangle subtends an angle of 60°; find a side of the triangle.

11. AB is a horizontal line whose length is 400 yards; from a point in the line between A and B a balloon ascends vertically, and after a certain time its altitude is taken simultaneously from A and B; at A it is observed to be 64° 15′; at B 48° 20′; find the height of the balloon when the observations are taken.

Given $L \sin 64° 15' = 9\cdot 9545793$ $L \sin 48° 20' = 9\cdot 8733352$
$L \sin 67° 25' = 9\cdot 9653532$ $\log_1 2\cdot 29149 = \cdot 4646213$
$\log_{10} 2 = \cdot 30103.$

XXXIII.

1. Show how to express in degrees, minutes, and seconds an angle whose circular measure is known.

Find, correct to three places of decimals, the radius of a circle in which an arc 15 inches long subtends at the centre an angle containing $71^\circ\ 36'\ 3{\cdot}6''$. ($\pi = 3{\cdot}1416$.)

2. Define the sine of an angle, and prove that—

$$\sin A = \sin(180^\circ - A) = \sin\{-(180^\circ + A)\}$$

* Write down formulæ, including all angles which satisfy—

(i.) $2 \sin A = 1$. (ii.) $2 \sin^2 A = 1$.

3. Prove that $\cos(A+B) = \cos A \cos B - \sin A \sin B$, and deduce expressions for $\cos 2A$, $\cos 3A$, in terms of $\cos A$.

4. Given $\cos A = {\cdot}28$, determine the value of $\tan \frac{A}{2}$, and explain fully the reason of the ambiguity which presents itself in your result.

5. Prove that—

(i.) $\tan\theta + \cot\theta = \sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}$.

(ii.) $\sec\theta - \tan\theta = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$.

* (iii.) $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$.

* (iv.) $\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}$.

6. State and prove the rules by means of which you can determine by inspection the integral part of the logarithm of any given number.

* Given $\log 4{\cdot}96 = {\cdot}6954817$, $\log 4{\cdot}9601 = {\cdot}6954904$, find the logarithms of 496010, ·000496, and 49600·25.

7. Show that in *any* plane triangle $a = b\cos C + c\cos B$.

If $c = \sqrt{2}$, $A = 117^\circ$, $B = 45^\circ$, find all the other parts of the triangle.

8. Find the greatest angle of the triangle whose sides are 50, 60, 70 respectively, having given—

$\log 6 = {\cdot}7781513$, $L \cos 39^\circ\ 14' = 9{\cdot}8890644$, diff. $1' = 1032$.

9. Express the area of a triangle in terms of one side and the two angles adjacent to it.

Two angles of a triangular field are $22\frac{1}{2}°$ and 45° respectively, and the length of the side opposite to the latter is a furlong. Show that the field contains exactly two acres and a half.

10. Find an expression for the diameter of the circle which touches one side of a triangle and the other sides produced.

If d_1, d_2, d_3 be the diameters of the three escribed circles of a triangle, show that $d_1d_2 + d_2d_3 + d_3d_1 = (a + b + c)^2$.

11. A man standing at a certain station on a straight sea-wall observes that the straight lines drawn from that station to two boats lying at anchor are each inclined at 45° to the direction of the wall, and when he walks 400 yards along the wall to another station he finds that the former angles of inclination are changed to 15° and 75° respectively. Find the distance between the boats, and the perpendicular distance of each from the sea-wall.

XXXIV.

1. Prove that the angle subtended at the centre of a circle by an arc equal in length to its radius is an invariable angle.

One angle of a triangle is 45°, and the circular measure of another is $1\frac{1}{2}$; find the third, both in degrees and in circular measure.

2. Define the secant of an angle, and show how your definition applies to angles between 180° and 270°.

If $\sec A = -2$, what two values between 0° and 360° may A have?

3. Obtain a formula embracing all the angles which have a given tangent.

Determine all the values of θ which satisfy the equation—

$$\sqrt{3}\tan^2\theta + 1 = (1 + \sqrt{3})\tan\theta.$$

4. Find an expression for $\tan 3A$ in terms of $\tan A$.

* Show also that $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$.

5. Prove that $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$; and that $\sin^2 30^\circ = \sin 18^\circ \sin 54^\circ$.

Show that in any circle the chord of an arc of 108° is equal to the sum of the chords of arcs of 36° and 60°.

6. Demonstrate the identities—

(i.) $\dfrac{(\operatorname{cosec} A + \sec A)^2}{\operatorname{cosec}^2 A + \sec^2 A} = 1 + \sin 2A.$

(ii.) $\sin 3A = 4 \sin A \sin (60^\circ + A) \sin (60^\circ - A).$

(iii.) $4 (\cot^{-1} 3 + \operatorname{cosec}^{-1} \sqrt{5}) = \pi.$

* 7. What are the advantages gained by the use of logarithms calculated to the base 10?

If $\log_{10} 2 = \cdot 30103$, find the logarithms of 5, $\dfrac{1}{125}$, $4\sqrt{\cdot 005}$, to base 10.

8. Prove that in any triangle (i.) $2bc \cos A = b^2 + c^2 - a^2$.

* (ii.) $\dfrac{1 + \cos (A - B) \cos C}{1 + \cos (A - C) \cos B} = \dfrac{a^2 + b^2}{a^2 + c^2}$.

9. If r_1 be the radius of a circle touching the side a of a triangle and the other two sides produced, show that—

$$r_1 \cos \frac{A}{2} = a \cos \frac{B}{2} \cos \frac{C}{2}.$$

* If a be the side of a regular polygon of n sides, and R, r, the radii respectively of its circumscribed and inscribed circles, prove that

$$R + r = \frac{a}{2} \cot \frac{\pi}{2n}.$$

10. Two sides of a triangle, which are respectively 250 and 200 yards long, contain an angle of 54° 36′ 24″; find the two other angles, having given

L cot 27° 18′ = 10·2872338 L tan 12° 8′ 50″ = 9·3329292
diff. for 1′ = 3100 log 3 = ·4771213.

11. The eye of a soldier in a straight trench of uniform depth is 2 feet above a level plain on which he sees two men standing in the same straight line as the trench; the parts of their bodies above the level of his eye subtending at it the angles $\tan^{-1} \cdot 0041\dot{6}$ and $\tan^{-1} \cdot 00\dot{4}$. On walking 100 feet towards them in the trench he notices that the height of one exactly hides that of the other; and, on approaching 596 feet 8 inches closer still he finds that the portion of the height of the nearer above the level of his eye subtends at it 45°. Find the heights of the men.

XXXV.

1. Define the "circular measure" of an angle, and show that, if with centre O any circle be described, cutting the bounding lines of an angle BOC in the points P, Q, the circular measure of the angle will be equal to $\frac{\text{arc}\,PQ}{OP}$.

* Find the number of seconds in the angle subtended at the centre of a circle, whose radius is 1 mile, by an arc $5\frac{1}{2}$ inches long.

2. Give accurate definitions of the sine and cosine of an angle, and prove that $\cos A = \sin(90° + A) = -\cos(180° + A)$.

Show that the sine will be algebraically less than the cosine for any angle between $(8n-3)\ 45°$ and $(8n+1)\ 45°$, where n is zero or any positive integer.

3. Find, geometrically, expressions for the sine and cosine of the sum of two angles, in terms of the sines and cosines of the angles themselves.

* The cosines of two angles of a triangle are $\frac{3}{5}$ and $\frac{12}{13}$ respectively. Find all the trigonometrical ratios of the third angle.

4. Express $\sin A$ and $\cos A$ in terms of $\tan\frac{A}{2}$.

* Prove that—

$$\tan\left(45° - \frac{A}{2}\right) + \tan\left(45° + \frac{A}{2}\right) = 2 \sec A.$$

5. Find the sine of 18°, and deduce the sine of 36°.

Two parallel chords of a circle, lying on the same side of the centre, subtend respectively 72° and 144° at the centre. Show that the distance between the chords is half the radius of the circle.

6. Find all the angles which satisfy the equation $2 \sin 2\theta = 3 \tan \theta$.

7. Prove that—

(i.) $\sec^6 \theta - \tan^6 \theta = 1 + 3 \tan^2 \theta . \sec^2 \theta$.

(ii.) $\text{vers}\,(270^\circ + A) . \text{vers}\,(270^\circ - A) = \cos^2 A$.

(iii.) $\dfrac{\sin A + 2 \sin 3A + \sin 5A}{\cos A - 2 \cos 3A + \cos 5A} = \dfrac{4 \sin A - 3 \operatorname{cosec} A}{4 \cos A - 3 \sec A}$.

8. Given $\log 2 = \cdot 30103$, $\log 3 = \cdot 4771213$, find L sin 45°, L sec 30°.

Also if L sin 15° = 9·4129962, what will be the value of L cos 15°?

9. Prove that in any triangle

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

If the angles adjacent to the base of a triangle are $22\frac{1}{2}^\circ$ and $112\frac{1}{2}^\circ$, show that the perpendicular altitude will be half the base.

10. Find an expression for the radius of the circle which touches one side of a triangle and the other two produced.

Show that the sum of the radii of the two escribed circles of a triangle, which touch the side c produced, is equal to to $c \cot \frac{C}{2}$.

11. Show how to solve a triangle by means of logarithms when the three sides are given.

Find the least angle of the triangle whose sides are 24, 22, 14, having given L tan 17° 33′ = 9·500042, diff. for 1′ = ·000439.

*. 12. A man walking along a straight road, which runs in a direction 30° east of north, notes when he is due south of a certain house. When he has walked a mile further, he observes that the house lies due west, and that a windmill on the opposite side of the road is N.E. of him. Three miles further on, he finds that he is due north of the windmill. Find the distance between the house and windmill, and show that the line joining them makes with the road an angle

$$\tan^{-1}\left(\frac{48 - 25\sqrt{3}}{11}\right).$$

XXXVI.

1. Assuming that the ratio of the circumference of a circle to the diameter is 3·1416, find to four places of decimals the value in degrees of the ordinary unit of circular measure.

If the radius of a circle is 4000 miles, determine the number of miles in an arc which subtends an angle at its centre whose circular measure is $\frac{\pi}{6}$.

2. Give a definition of the sine of an angle which will apply to angles of any magnitude. Carefully prove that, whatever be the magnitude of A, $\sin(90^\circ + A) = \cos A$, and $\cos(90^\circ + A) = -\sin A$.

3. If A and B be each less than 90°, but their sum greater, draw the appropriate figure, and give a geometrical proof that—

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

Deduce the expressions for $\sin(A+B)$, and show that—

$$\cos^4 A = 8\cos^4 A - 8\cos^2 A + 1.$$

4. Prove that—

$$\text{(i.)}\ \tan A = \frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A}.$$

$$\text{(ii.)}\ \frac{\cos(A+45^\circ)}{\cos(A-45^\circ)} = \sec 2A - \tan 2A.$$

$$\text{(iii.)}\ \cot^{-1}\frac{1}{3} = \cot^{-1} 3 + \cot^{-1}\frac{3}{4}.$$

5. Find sin 150°, sin 225°, sin 15°, sin 37° 30′ without reducing the quadratic surds.

6. Obtain an expression for all the angles whose cosine is equal to a given value.

If $3\cos^2\theta + 2\sqrt{3}\cos\theta = 5\frac{1}{4}$, find the general value of θ.

* Explain why it is that exactly the same series of angles are given by the two equations—

$$\theta + \frac{\pi}{4} = n\pi + (-1)^n\frac{\pi}{6}\ ;\ \text{and}\ \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$

7. Find expressions for the radii of the inscribed and escribed circles of a triangle, each expression involving one side and functions of the halves of the angles of the triangle.

If r, r_1, r_2, r_3 be the radii, and a, b, c the sides, prove that—

$$\frac{r_1 - r}{a} + \frac{r_2 - r}{b} = \frac{c}{r_3}.$$

8. Show that in any triangle $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$.

* Obtain an expression for the area in terms of the sides, and if the lengths of the sides be 242, 1212, and 1450 yards, show that the area is 6 acres.

* 9. Assuming the formula—

$$\log\sin(\theta+h) - \log\sin\theta = \mu h\cot\theta - \frac{\mu h^2}{2}\operatorname{cosec}^2\theta,$$

where h is the circular measure of a small angle, and μ the modulus of the logarithms, and powers of h above the second are neglected, explain when the principle of proportional parts will fail in its application to tables of logarithms of sines.

If L sin 15° 30′ be 9·426899, and the difference for 1′ be ·000455, find accurately L sin 15° 30′ 36″; and the angle whose tabular logarithmic cosine is equal to 9·427263.

Given $\log_{10} 2 = {\cdot}301030$ and $\log_{10} 3 = {\cdot}477121$, find $\log_{10} 3\frac{6}{25}$.

* 10. If in any triangle b, c, and A be given, prove that

$$\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2};$$

hence obtain the formula—

$$\tan\frac{1}{2}(B-C) = \tan^2\frac{\phi}{2}\cot\frac{A}{2},$$

and explain the advantage of such a formula.

* If b be to c as 11 to 10, and $A = 35° 25'$, apply the above formula to find B and C, having given

$$\log 1{\cdot}1 = {\cdot}041393$$

L tan 12° 18′ 36″ = 9·338891   L cos 24° 37′ 12″ = 9·958607

L tan 8° 28′ 56·5″ = 9·173582   L cot 17° 42′ 30″ = 10·495800.

11. From a house on one side of a street observations are made of the angle subtended by the height of the opposite house: first from the level of the street, in which case the angle is $\tan^{-1}(3)$; and afterwards from two windows one above the other, from each of which the angle is found to be $\tan^{-1}(-3)$. The height of the opposite house being 60 feet, find the height of each of the two windows above the street.

XXXVII.

1. State briefly the advantages or disadvantages of the English or French methods, respectively, of measuring angles. If an angle be expressed in French minutes, show that it will be transferred to English minutes when multiplied by ·54. Verify this by first expressing an angle of an equilateral triangle in French measure. Divide the English angle 77° into two parts, so that the number of English seconds in one part may equal the number of French seconds in the other part.

2. Define the sine and cosine of an angle. Prove that the sine of any angle is the cosine of its complement. If A be greater than two, and less than three right angles, represent geometrically the complement of $A-180°$.

If n be a positive whole number, show that the angles $(2n \,.\, 180° + A)$ and $(2n+1)\,180° - A$ have the same sine as A. Given the sine of an angle, find the tangent. Prove $\tan 60° = \sqrt{3}$.

* 3. Without assuming the formula for $\sin(A+B)$, prove geometrically $\sin 2A = 2\sin A \cos A$, A being less than 45°. Obtain the equation—

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}, \text{ and hence prove } \tan\frac{45°}{2} = \sqrt{2}-1.$$

* 4. Express $\sin A$ in terms of $\sin 2A$. In the general expression, how many values of $\sin A$ are thus obtained? Show in any particular case how the correct value is to be selected. Assuming the equation $\cos 3A = 4(\cos A)^3 - 3\cos A$, find $\sin 18°$, and hence exhibit the true value of $\sin 9°$ without reducing the surds.

5. Prove—

(i.) $(\cos A)^4 - (\sin A)^4 = \cos 2A$.

(ii.) $\sin A + \sin(72° + A) + \sin(36° - A) = \sin(72° - A) + \sin(36° + A)$.

(iii.) $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2} = 120°$.

6. Given the three sides of a triangle, find the cosine of one of the angles, and hence express the sine of half that angle in terms of the sides and in a form adapted to logarithmic computation. If a, b, c be the sides subtending the angles A, B, C respectively of the triangle ABC, prove—

$$c=\frac{(a-b)\cos\frac{C}{2}}{\sin\frac{A-B}{2}}.$$

7. Find the radius of the escribed circle of the triangle ABC, when the circle touches the side BC and the sides AB and AC produced.

8. In a triangle ABC, given $BC=2AC$, $A=3B$, find the angles of the triangle, and the ratio of AB to BC and AC.

9. State the property which furnishes the principle for constructing the proportional parts in the ordinary tables of logarithms.

Given $\log_{10} 6{\cdot}0389 = {\cdot}7809578$, $\log_{10} 6{\cdot}0390 = {\cdot}7809650$, calculate the corresponding proportional parts, and find the number of which ·7809601 is the logarithm.

Given $\log_{10} 2 = {\cdot}3010300$, find $\log_{10} 250$, and the tabulated logarithms of sin 30° and cosec 30°.

10. Explain when the solution of a triangle is said to be ambiguous.

Supposing the data for the solution of a triangle to be as in the two following cases (α), (β), point out in each case whether the solution will be ambiguous or unambiguous.

Find the angle C in the case not ambiguous, and the third side of the obtuse-angled triangle in the ambiguous case.

(α.) $A=30°$ $AB=250$ feet $BC=125$ feet
(β.) $A=30°$ $AB=250$ feet $BC=200$ feet
$L \sin 38° 41' = 9{\cdot}7958800$ $L \sin 8° 41' = 9{\cdot}1789001$.

[*Note.*—The logarithms of Question 9 to be used, when required.]

11. A church tower BCD, with a spire above it, stands on a horizontal plane, B being a point in its base, and C being 9 feet vertically above B. The height of the tower is 289 feet, and of the spire 35 feet; from the extremity A of a horizontal line BA, it is found that the angle subtended by the spire is equal to the angle subtended by BC. Prove that $BA=180$ feet nearly.

XXXVIII.

1. In plane trigonometry state how a right angle is usually divided by English and French mathematicians respectively. If the circumferences of the quadrants of two circles be divided similarly to the right angles they subtend, what would be the radius of a circle divided according to the French scale, in which the length of the arc of one grade would be equal to the length of the arc of one degree on a circle whose radius was 18 feet? Express in the French scale (1) the sum of the angles of a quindecagon; (2) one of the angles when the quindecagon is equiangular.

2. According to the ratio definition, point out which of the elementary trigonometrical functions are never less than unity, and which may be either less or greater than unity.

Prove $(\sin A)^2 + (\cos A)^2 = 1$, and express the numerical values of $\sin 135°$ and $\tan 150°$ with their proper signs.

3. If $(A+B)$ is an angle in the first quadrant, obtain by means of a geometrical figure the formula $\cos(A+B) = \cos A \ \cos B - \sin A \ \sin B$, and deduce from it the expression for $\cos(A-B)$.

Find $\sec(A+B)$ in terms of $\sec A$ and $\sec B$, and prove—

$$\sec 105° = -\sqrt{2}\,(1+\sqrt{3}).$$

4. Prove—

(i.) $\sin 3A = 3\sin A - 4(\sin A)^3$.

* (ii.) $\sin 54° = \frac{\sqrt{5}+1}{4}$.

(iii.) $\frac{\sin(A-C)}{\cos A \ \cos C} + \frac{\sin(B-A)}{\cos B \cos A} + \frac{\sin(C-B)}{\cos C \cos B} = 0.$

5. Prove that in every circle the angle subtended by an arc equal to the radius is an invariable angle. Express that angle in degrees and decimals of degrees, and find the circular measure of one minute. On a circle 10 feet in radius it was found that an angle of $22° \, 30'$ was subtended by an arc 3 feet $11\frac{1}{8}$ inches in length; hence calculate, to four decimal places, the numerical ratio of the circumference of a circle to its diameter.

6. If a, b, c be the sides subtending respectively the angles A, B, C of a plane triangle:

Prove—

$$\text{(i.)}\quad \frac{\sin A}{\sin(A+B)}=\frac{a}{c}.$$

$$\text{(ii.)}\quad a=(b-c)\sec\theta,\ \text{if}\ \tan\theta=\frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2}.$$

$$\text{(iii.)}\quad \tan^{-1}\frac{a}{b+c}+\tan^{-1}\frac{b}{a+c}=45^\circ\ \text{when}\ C\ \text{is a right angle.}$$

7. If (r) be the radius of the circle inscribed in the triangle ABC,

$$\text{Prove (i.)}\quad r=\frac{2\ \text{area}}{a+b+c}.$$

$$\text{(ii.)}\quad r=\frac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}}.$$

8. In a right-angled triangle ABC, C being the right angle, find AB, if $B=30^\circ$ and $BC=100$ feet.

$A'B'C'$ is also a right-angled triangle, C' the right angle and $B'=30^\circ$, find $A'C'$ when the area of the triangle $A'B'C'$ is three times the area of ABC.

9. When two sides and an included angle of a triangle are given, investigate a formula for determining the other two angles. Show that for this determination it is not necessary to know the absolute lengths of the two sides, provided their ratio is given.

Ex. The included angle is 70° 30′, the ratio of the containing sides is 5 : 3, find the other angles.

$$\text{Given } \log_{10} 2 = \cdot 3010300$$
$$\text{L}\cot 35^\circ\,15' = 10\cdot 1507464$$
$$\text{L}\tan 19^\circ\,28'\,50'' = 10\cdot 5486864.$$

10. A tower stood at the foot of an inclined plane whose inclination to the horizon was 9°; a line was measured straight up the incline from the foot of the tower of 100 feet in length, and at the upper extremity of this line the tower subtended an angle of 54°; find the height of the tower.

[*Note.*—For sin 54°, see Question 4.]

11. Show that the logarithms of proper fractions are negative. Express the true value of the logarithm of $\frac{1}{200}$ to the base 10. How would it be expressed with a negative characteristic? Since $\sin 30° = \frac{1}{2}$, explain why the logarithm of sin 30° is tabulated 9·6989700. Given that L tan 38° 16′ is tabulated 9·8969714, determine L cot 38° 16′.

* Given L tan 38° 16′ 10″ = 9·8970147, L tan 38° 16′ 20″ = 9·8970580, find the angle whose logarithmic tangent is 9·8970365, log 2 = ·3010300.

XXXIX.

1. Examine the rule for transferring an angle expressed in French measure to the English scale. Express the angle of an equilateral and equiangular pentagon in the French scale, and thence determine its measure in the English scale.

2. Define as ratios the sine and tangent of an angle, and show that any proper fraction, positive or negative, may be the sine of an angle, and any number, positive or negative, may be the tangent of an angle.

Prove $\cos(90° - A) = \sin A = -\sin(-A)$.

3. If $A + B$ be less than a right angle, prove—

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

Deduce directly from this formula $\cos 2A = \cos^2 A - \sin^2 A$.

* Express $\frac{1}{2}[\cos(3A + 2B) + \cos(5A - 4B)]$ as the product of two cosines.

4. Find the sine and cosine of 30°; and prove $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$, and $\sin 105° + \cos 105° = \cos 45°$.

5. Prove—

$$\text{(i.)}\ \tan A = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}.$$

$$\text{(ii.)}\ 1 + \tan 2A \tan A = \sec 2A.$$

$$\text{(iii.)}\ \cos 36° = \frac{\sqrt{5}+1}{4}.$$

6. If π ($=3{\cdot}1416$) be the ratio of the circumference of a circle to its diameter, of what angle is $\frac{\pi}{6}$ the circular measure? Express in degrees the angular unit of circular measure; and show that this angle is subtended by an arc equal in length to the radius of the circle. Express in circular measure the least angle of an isosceles triangle in which the vertical angle is half of each of the angles at the base.

* 7. An observer on the top of a cliff 200 feet above the sea level, observes the angles of depression of two ships at anchor in the same vertical plane with the observer to be 45° and 30° respectively; find the distance of the ships from each other.

8. When is the solution of an oblique-angled triangle said to be ambiguous? Given $A=30°$, $BC=100$, $AB=100\sqrt{3}$, solve both the triangles to which the data belong.

9. Given the three sides of a plane triangle, find an expression for the area of the triangle.

* Apply it to determine the area of the triangle whose three sides are 1·9, 1·2, 1·5 respectively.

10. In any triangle whose sides a, b, c, subtend the angles A, B, C when $s=\frac{a+b+c}{2}$, prove—

* (i.) $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$;

(ii.) $s(s-a)=(s-b)(s-c)$, if A is a right angle.

* 11. Given L sin 73° 20′ 10″ = 9·9813671

L sin 73° 20′ 20″ = 9·9813734

find L sin 73° 20′ 15·7″.

12. Find an expression for the radius (R) of the circle circumscribing a triangle. If r be the radius of the inscribed circle, show that—

$$\frac{r}{R}=4\left\{\frac{s}{a}-1\right\}\left\{\frac{s}{b}-1\right\}\left\{\frac{s}{c}-1\right\}.$$

XL.

1. Explain the English and French methods of measuring angles. Express according to the French method an angle containing $4\frac{1}{2}$ English minutes.

2. Define the sine of an angle, and find the greatest and least values which a sine can have.

Prove that $\sin A = -\sin (180^\circ + A)$.

3. Show that $\cos (A - B) = \cos A \cos B + \sin A \sin B$.

* Express $2 \sin 5A \sin 9A$ as the difference of two cosines.

4. Find the tangents of 45°, 60°, and 15°.

Write down a general formula for all angles whose tangent $= -1$.

5. Prove that—

(i.) $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$.

(ii.) $\cot A + \operatorname{cosec} A = \cot \dfrac{A}{2}$.

(iii.) $\sin 18^\circ + \sin 30^\circ = \sin 54^\circ$.

6. Show that the circular measure of an angle less than a right angle is always intermediate in value between its sine and its tangent.

Calculate roughly the distance at which a globe $5\frac{1}{2}$ inches in diameter will subtend an angle of $6'$. $\left(\pi = \frac{22}{7}.\right)$

7. One side of a triangle is double of another, and the included angle is a right angle.

Find the other angles, having given

$\log 2 = \cdot 30103$, $L \cot 26^\circ 34' = 10\cdot 3009994$, diff. for $1' = \cdot 0003159$.

8. Prove that the sides of any triangle are proportional to the sines of the opposite angles.

The altitude of a certain rock is observed to be 47°, and, after walking towards it 1000 feet up a slope inclined at 32° to the horizon, the observer finds that the altitude is 77°. Find the vertical height of the rock above the first point of observation, having given $\sin 47^\circ = \cdot 73135$.

9. Find expressions for r, R, the radii of the inscribed and circumscribed circles of a triangle, and show that

$$* \; r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}.$$

10. Show that in any triangle—

(i.) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$

(ii.) $\tan\frac{3A}{2} = \frac{\sin 3B - \sin 3C}{\cos 3C - \cos 3B}.$

11. The hypothenuse and shortest side of a right-angled triangle are 5 feet and 3 feet respectively. Find the length of the perpendicular from the right angle upon the hypothenuse, and show that it is inclined at an angle $\sin^{-1}\frac{7}{25}$ to the straight line drawn from the right angle to the middle point of the hypothenuse.

XLI.

1. Show how to transform the circular measure of an angle into degrees.

Find the unit, when the circular measure of 120° is represented by $\frac{15}{16}$.

2. Show that $\sin(90° + A) = \cos A = \sin(90° - A)$; and find the value of the trigonometrical functions for an angle of 2670°.

3. Prove that—

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ when A is a positive angle between $\frac{\pi}{4}$ and $\frac{\pi}{2}$, and B a negative angle between $-\pi$ and $-\frac{5\pi}{4}$.

Find the values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in terms of $\cos A$.

4. Obtain the values of sin 18° and cos 75°.

* Show that $\frac{1 + \cot 60°}{1 - \cot 60°} = \left\{\frac{1 + \cos 30°}{1 - \cos 30°}\right\}^{\frac{1}{2}}.$

5. Prove the following identities—

(i.) $\cos 3A = 4\cos^3 A - 3\cos A$.

* (ii.) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4} = \sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1} 3$.

6. In any triangle ABC—

(i.) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$.

(ii.) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$.

7. $ABCD$ is a quadrilateral whose opposite sides, AB and CD, meet at right angles. Show that—

$$\cot DAB \cot ABC = \cot CAB \cot ABD.$$

8. If r, r_a, r_b, r_c are the radii of the circle inscribed in, and of the circles escribed to the triangle ABC, then

$$r_a \cot\frac{A}{2} = r_b \cot\frac{B}{2} = r_c \cot\frac{C}{2} = r \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

9. Explain geometrically the meaning of the ambiguous case in the solution of triangles.

* If a, b are the two given sides, $a<b$, A the given angle, and C, C' the two values of the ambiguous angle, then

$$\cos A \cos B = \sin\frac{C+C'}{2}\sin\frac{C-C'}{2}.$$

* 10. Show that in general the increment of the tabular logarithmic cosine of an angle is approximately proportional to the increment of the angle; and being given

$$\text{L}\cos 55^\circ\ 11' = 9{\cdot}7565999$$
$$\text{L}\cos 55^\circ\ 12' = 9{\cdot}7564182$$

find the tabular logarithm of the cosine of 55° 11′ 12″.

* 11. The angular elevation of a tower at a place due south of it is 45°, and at another place due west of the former the elevation is 15°. Show that the height of the tower is

$$\frac{a}{2}\left[3^{\frac{1}{4}} - 3^{-\frac{1}{4}}\right]$$

where a is the distance between the two places.

XLII.

1. Show how to express (1) in grades, (2) in circular measure, an angle whose magnitude is given in degrees.

The angles of a triangle are in the ratio 1 : 2 : 3. Express their magnitudes in each of the three systems of angular measurement.

2. Show that $\sin(180^\circ - A) = \sin A$, and that $\cos(180^\circ - A) = -\cos A$.

If $\cot \frac{A}{2} = 1 + \sqrt{2}$, find $\cos A$.

3. Establish the formula—

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

and show that

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}.$$

4. Show that—

(i.) $\operatorname{cosec} A = \cot A + \tan \frac{A}{2}$.

(ii.) $1 = \operatorname{cosec}^2 A - \cot^2 A$.

(iii.) $\tan A = (1 + \sec A) \tan \frac{A}{2}$.

* 5. Show that $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$;

and that $2 \cot^{-1} x = \operatorname{cosec}^{-1} \frac{1+x^2}{2x}$.

* 6. Solve the equation $\operatorname{cosec} \theta + \cot \theta = \sqrt{3}$.

7. If A, B, C be the angles of a triangle, and a, b, c the sides opposite to them respectively, show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

and each $= \frac{1}{2abc} \sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}$

8. If in the triangle ABC, C is a right angle, show that—

$$\tan(A \backsim B) = \frac{a^2 \backsim b^2}{2ab}.$$

9. Find an expression for the radius of the circle inscribed in a triangle.

The points where the circle inscribed in a triangle ABC touches the sides are joined. Show that the sides of the triangle thus formed are proportional to $\cos\frac{A}{2}$, $\cos\frac{B}{2}$, $\cos\frac{C}{2}$ respectively.

19. In the triangle ABC, $b=5$, $c=4$, $A=60°$. Find B and C, having given

$$\begin{aligned} \log 3 &= \cdot 477121 \\ \text{L}\tan 10° \, 53' &= 9\cdot 283907 \\ \text{L}\tan 10° \, 54' &= 9\cdot 284525. \end{aligned}$$

11. Two circles, radii a and b, touch one another externally. Show that the angle between their common tangents is

$$2\tan^{-1}\frac{1}{2}\left(\sqrt{\frac{a}{b}} \backsim \sqrt{\frac{b}{a}}\right).$$

XLIII.

1. Show that the angle subtended at the centre of a circle by an arc equal to the radius is invariable.

Taking 22 : 7 as the ratio of the circumference of a circle to its diameter, and assuming the diameter of the Earth to be 8000 miles, find approximately the difference in latitude of two places, one of which is 100 miles north of the other.

* 2. Find an expression for $\sin\frac{A}{2}$ in terms (1) of $\sin A$, (2) of $\cos A$, and explain why more than one value appears.

Find sin 15° and cos 165°.

3. Show that—

(i.) $\cot A - \tan A = 2\cot 2A$.

(ii.) $\tan(A+B)\tan(A-B) = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}$.

* 4. Find an expression for all the angles which have a given tangent.

Show that $\tan^{-1}(\cot A) - \tan^{-1}(\tan A) = n\pi + \frac{\pi}{2} - 2A$.

5. Solve the equation $\tan^2\theta = 3 \operatorname{cosec}^2\theta - 1$.

6. In a triangle ABC, A is a right angle and $B = 60°$. Show that $2a = c + b\sqrt{3}$.

7. Show that in every triangle—

$$\text{(i.)}\ \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$*\ \text{(ii.)}\ \frac{a\cos B - b\cos A}{c} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C}.$$

8. Find an expression for the area of a triangle.

In an isosceles right-angled triangle a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Show that it divides the angle into parts whose cotangents are 2 and 3.

9. Find an expression for the radius of the circle inscribed in a given triangle.

Two sides of a triangle are a and $2a$, and they include a right angle. If r_1, r_2 be the radii of the escribed circles touching these sides, show that $r_1 r_2 = a^2$.

10. Two sides and the angle opposite to one of them being given, explain by means of a figure why more than one triangle may in some cases be found having the given parts.

In a triangle $a = 5$, $b = 4$, $A = 45°$. Find the other angles, having given

$\log 2 = \cdot 3010300$ $\quad$ $L \sin 33°\ 29' = 9\cdot 7520507$

$L \sin 33°\ 30' = 9\cdot 7530993$.

11. If θ be the acute angle between two adjacent sides of a rhombus, show that—

$$\theta = \cos^{-1}\frac{x^2 \backsim y^2}{x^2 + y^2}$$

where x, y are the lengths of the diagonals.

XLIV.

1. Find the circular measure of an angle of A degrees.

Find the radius of a globe such that the distance measured upon its surface between two places on the same meridian whose latitudes differ by $1^\circ\ 10'$ may be one inch, assuming the unit of circular measure to be $\frac{7}{11}$ of a right angle.

2. Trace the changes in sign and magnitude of $\cos A$ through the four quadrants; and deduce a general expression for all angles having the same cosine.

If $\cos p\theta + \cos q\theta = 0$, prove that the different values of θ form two arithmetical progressions in which the common differences are $\frac{2\pi}{p+q}$ and $\frac{2\pi}{p \backsim q}$ respectively.

3. Prove that $\sin(A+B) = \sin A \cos B + \cos A \sin B$, pointing out any restrictions in the value of A and B assumed in your proof.

If $x \cos\beta + y \cos\alpha = z, \quad x \sin\beta - y \sin\alpha = 0,$

prove that $\frac{x}{\sin\alpha} = \frac{y}{\sin\beta} = \frac{z}{\sin(\alpha+\beta)}$.

4. Prove that $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$, and determine the limits to the value of A, in order that the lower sign may be taken.

Prove that $2\cos^2 A \cos^2 B + 2\sin^2 A \sin^2 B = 1 + \cos 2A \cos 2B$, and $\tan 2A = (1 + \sec 2A)\tan A$.

5. Find $\sin 3A$, and apply the result to determine $\sin 18^\circ$.

6. Explain the notation $\tan^{-1} m$.

If $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$, prove that $a+b+c = abc$.

7. In any triangle ABC whose sides are a, b, c, prove that

$$\sin A : \sin B : \sin C :: a : b : c.$$

If in such a triangle $\frac{\tan A - \tan B}{\tan A + \tan B} = \frac{c-b}{c}$; prove that $A = 60^\circ$.

8. Given two sides and the difference of the angles opposite to them; solve the triangle. If the sides be 4 and 5, and the cosine of the difference of the opposite angles be $\frac{31}{32}$, prove that the third side will be 6.

9. Find the radius of the circle circumscribing a given triangle.

If R and r be the radii of the circumscribing and inscribed circles, and a, b, and c the sides of the triangle, prove that

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}.$$

10. If $a=3$, $b=1$, $C=53^\circ\ 7'\ 48''$, find c, without determining A and B with the aid of the following logarithms :—

$\log 2 = \cdot 3010300$

$\log 25298 = 4\cdot 4030862$

$\log 25299 = 4\cdot 4031034$

$L \cos 26^\circ\ 33'\ 54'' = 9\cdot 9515452$

$L \tan 26^\circ\ 33'\ 54'' = 9\cdot 6989700.$

11. Find the area of a regular polygon of n sides inscribed in a circle of radius R.

Find, by the aid of logarithmic tables, the area of a regular polygon of eleven sides, inscribed in a circle of radius one foot, to three places of decimals.

XLV.

1. Define the circular measure of an angle.

Find the number of degrees subtended at the centre of a circle by an arc $= \cdot 357$ radius, taking π to be $3\cdot 1416$.

2. Define the principal trigonometrical ratios, and express each of them in terms of the tangent.

If $1+\sin^2\theta=3\sin\theta\cos\theta$, find $\tan\theta$.

3. Write down a formula for all angles having the same sine.

$$\text{If } \frac{\sin^4\theta}{\sin^2 a}+\frac{\cos^4\theta}{\cos^2 a}=1;$$

prove that the general value of θ is $n\pi \pm a$ where n is any integer.

4. Prove geometrically that—

$$\sin(A-B)=\sin A\cos B-\cos A\sin B,$$

where A is $>$ B and A and B are each less than 90°.

$$\text{If } \quad a\sin A+b\sin B+c\sin C=0,$$

$$\text{and } a\cos A+b\cos B+c\cos C=0,$$

$$\text{prove that } \frac{\sin(B-C)}{a}=\frac{\sin(C-A)}{b}=\frac{\sin(A-B)}{c}.$$

5. Find $\sin\frac{A}{2}$ in terms of $\sin A$. Apply the formula to find $\sin 15°$.

6. Find $\sin 3A$ and $\cos 3A$, and prove that—

$$\cos 3A - \sin 3A = (\cos A + \sin A)\,(1 - 2\sin 2A).$$

Prove also that—

$$\cos(45° - A)\cos(45° - B) - \cos(45° + A)\cos(45° + B) = \sin(A + B).$$

7. Explain the notation $\tan^{-1} m$.

* Given $2\tan^{-1}\cos\theta = \tan^{-1} 2\operatorname{cosec}\theta$, find θ.

8. Prove that in any triangle ABC,

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

If $a = 1·56234$, $b = ·43766$, $C = 58°\ 42'\ 6''$, find A and B by the aid of the following logarithms—

$\log 56234 = 4\frac{3}{4}$ $\log\cot 29°\ 21' = ·250015$

$\log\cot 29°\ 22' = ·249715.$

9. Find an expression for the radius of a circle inscribed in a given triangle.

Prove that if this radius be r, and if p_1, p_2, p_3, be the perpendiculars drawn from the angles upon the opposite sides respectively, then—

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

10. A person standing at a point A, due south of a tower built on a horizontal plane, observes the altitude of the tower to be 60°. He then walks to a point B, due west from A, and observes the altitude to be 45°, and again at the point C in AB produced he observes the altitude to be 30°. Prove that B is midway between A and C.

11. If D be the middle point of the side BC of the triangle ABC, and if S be the area of the triangle, prove that $\cot ADB = \frac{AC^2 - AB^2}{4S}$.

12. Find the area of a regular polygon of n sides, the length of each side being a.

If the length of each side of a regular heptagon be 1 foot, prove, by means of logarithmic tables, that the area of a heptagon in square feet (to 3 places of decimals) will be 3·634, and find the difference between the area of the heptagon and that of the circumscribing circle, taking π to be 3·1416.

XLVI.

1. Define the secant, cosecant, tangent, and cotangent of an angle, and prove the relations $\sec^2 A = 1 + \tan^2 A$, $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

Find the values of sec 45° and of tan 45°.

2. Trace the changes in the sign and value of the cosine of an angle as the angle changes from 0° to 360°.

Find an expression for all the angles which have the same cosine as a given angle.

What is the general value of θ when

$$\text{(i.)}\ \cos\theta = \pm\frac{1}{\sqrt{2}}, \qquad \text{(ii.)}\ \cos 4\theta = \pm\frac{1}{\sqrt{2}}\ ?$$

3. Having given $\dfrac{\sin A}{\sin B} = \sqrt{\dfrac{2}{3}}$ and $\dfrac{\tan A}{\tan B} = \dfrac{1}{\sqrt{3}}$, find the value of $\sin A$ and $\sin B$; and if A and B be two angles of a triangle ABC, find the three angles.

4. Taking the case in which A is greater than B and less than a right angle, prove that $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

Having given $\cos A = \dfrac{3}{5}$ and $\cos B = \dfrac{4}{5}$, find the value of $\cos\dfrac{1}{2}(A - B)$, prove that if x and y be each less than unity,

$$\sin^{-1}x - \sin^{-1}y = \cos^{-1}\sqrt{1 - x^2 - y^2 + x^2y^2} + xy.$$

5. Prove the formula—

$$\text{(i.)}\ \frac{\sin A + \sin 2A}{\cos A + \cos 2A} = \tan\frac{3A}{2}.$$

$$\text{(ii.)}\ \sin 3A = 3\sin A - 4\sin^3 A.$$

$$\text{(iii.)}\ \sin(A + 45^\circ) + \sin(A + 135^\circ) = \sqrt{2}\cos A.$$

* 6. If $A + B + C = 180^\circ$, prove that—

$$\text{(i.)}\ \sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C.$$

$$\text{(ii.)}\ \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

7. Show that, in any triangle, the sides are proportional to the sines of the opposite angles. Having given that

$$a=3795 \text{ yards}, B=73^\circ\ 15'\ 15'', C=42^\circ\ 18'\ 30'',$$

find, by help of logarithmic tables, the other sides of the triangle ABC.

* 8. If R be the radius of the circle circumscribing the triangle ABC, and if p_1, p_2, p_3, be the distances from the sides of the centre of the circle, show that $2R=\dfrac{a}{\sin A}$, and that $\dfrac{a}{p_1}+\dfrac{b}{p_2}+\dfrac{c}{p_3}=\dfrac{abc}{4p_1p_2p_3}$.

9. A statue on the top of a pillar, situated on level ground, is found to subtend the greatest angle a at the eye of an observer, when his distance from the pillar is c feet; prove that the length of the statue is $2c \tan a$ feet; and, having given the height above the ground of the observer's eye, find the height of the pillar.

10. Having given the radius r of a circle inscribed in a regular polygon of n sides, find the area of the polygon, and hence prove that the area of a circle of radius r is πr^2.

XLVII.

1. Define the degree and the unit of circular measure, and show how to pass from one of these units to the other.

If the radius of a circle be 25 feet, find to four decimal places the length of the arc subtending an angle of 3°, taking π to be 3·1416.

2. Define the principal trigonometrical ratios, and trace the changes in sign and magnitude of $\dfrac{\sin 3\theta}{\cos 2\theta}$ as θ varies from 0 to $\dfrac{\pi}{2}$.

3. Prove that $\sin(A+B)=\sin A \cos B+\cos A \sin B$ where A and B are positive angles each less than 90°.

* Assuming that the formula holds for all values of the angles, deduce that for $\cos(A-B)$.

4. Find $\cos 3A$ and $\tan 3A$ in terms of $\cos A$ and $\tan A$ respectively.

5. Prove that for all values of A—

* (i.) $2 \operatorname{cosec} 4A + 2 \cot 4A = \cot A - \tan A$.

(ii.) $\dfrac{\cos A + \cos 3A}{\cos 3A + \cos 5A}=\dfrac{1}{2\cos 2A - \sec 2A}$.

6. Prove that—

$$\tan^{-1}m+\tan^{-1}n=\cos^{-1}\frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}.$$

7. Find sin 18°.

If $\cos A=\tan B$ and $\cos B=\tan A$;

Prove that $\cos A=2\sin 18°$.

8. Prove the formula—

$$a^2=b^2+c^2-2bc\cos A,$$

and apply it to prove that if the straight line which bisects the vertical angle of a triangle also bisects the base, then the triangle must be isosceles.

9. Find the area of a triangle in terms of the sides.

10. Find the radius of the circle which touches one side of a triangle and the two other sides produced.

If ABC and DEF be two triangles with the same perimeter, prove that the radius of the escribed circle touching the side BC of the first is to the radius of the escribed circle touching the side EF of the second as $\tan\frac{A}{2}$ is to $\tan\frac{D}{2}$.

11. Given two sides b and c, and the included angle A of a triangle, find a formula for determining the side a, without the previous determination of the angles B and C, adapted to logarithmic computation.

Apply to the case of $b=354$ yards, $c=426$ yards, $A=49°\ 16'$.

12. A building on a square base $ABCD$ has two of its sides AB and CD parallel to the bank of a river.

An observer standing on the river's bank in the same straight line with DA finds that the side AB subtends, at his eye, an angle of 45°.

Having walked (a) yards along the bank he finds that the side DA subtends an angle whose sine is $\frac{1}{3}$.

Prove that the length of each side of the building is $\frac{a}{\sqrt{2}}$ yards.

[Logarithm tables required.]

XLVIII.

1. Define the unit of circular measure.

Show how to transform from degrees to circular measure. What would be the unit of circular measure if an angle of 120° expressed in it became $\frac{3}{4}$?

2. Define the principal trigonometrical ratios, and express each of them in terms of the tangent.

If the cosecant of an angle between 90° and 180° be $\frac{2}{\sqrt{3}}$, find the secant, and if the cosine of an angle between 540° and 630° be $-\frac{1}{2}$, find the cosecant.

3. Prove that—

$$\sin(A+B) = \sin A \cos B + \cos A \sin B;$$

and deduce the value of $\tan(A+B)$ in terms of the tangents of A and B.

4. Find $\sin 3A$, and apply to determine the values of $\sin 18°$ and $\cos 36°$.

5. Prove that for all values of A—

$$(\sin 2A)^2 = 2\cos^2 A\,(1-\cos 2A);$$

and if A, B, and C be the angles of a triangle prove that—

$$\sin A \ \sin B + \cos^2\left(A+\frac{C}{2}\right) = \cos^2\frac{C}{2}.$$

* 6. Eliminate θ between the equations—

$$x = 2a\cos\theta\cos 2\theta - a\cos\theta,$$
$$y = 2a\cos\theta\sin 2\theta - a\sin\theta.$$

7. What is the meaning of $\sin^{-1} m$?

Prove that—

$$\sin^{-1}\frac{2mn}{m^2+n^2} + \sin^{-1}\frac{m^2-n^2}{m^2+n^2} = \frac{\pi}{2} \text{ for all values of } m \text{ and } n.$$

8. In any triangle prove that the sides are proportional to the sines of the opposite angles.

* If α, β, γ be the perpendiculars from the angles of the triangle ABC upon the opposite sides a, b, and c, prove that—

$$a\sin A + b\sin B + c\sin C = 2(\alpha\cos A + \beta\cos B + \gamma\cos C).$$

9. Prove that in any triangle ABC whose sides are a, b, and c—

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, and hence show that in such a triangle—

$$\frac{\cos\frac{A}{2}}{\sqrt{a(s-a)}} = \frac{\cos\frac{B}{2}}{\sqrt{b(s-b)}} = \frac{\cos\frac{C}{2}}{\sqrt{c(s-c)}}.$$

10. Show how to solve a triangle when two sides and the included angle are given.

If two of the sides be $a \sin a$ and $a \sin(120° - a)$ and the included angle be 60°, solve the triangle.

11. The angles of elevation of a balloon are observed from two stations a mile apart and from a point half way between them to be 60°, 30°, and 45° respectively. Prove that the height of the balloon in yards is $440\sqrt{6}$.

12. The adjacent sides AC and CB of the base of a rectangular building are 40 and 30 yards respectively, and the angles AOC and BOC subtended by them at a point O on the horizontal plane are each equal to $\tan^{-1}\frac{3}{4}$. Prove that the angle ACO is equal to $\pi - \tan^{-1}\frac{24}{25}$.

Hence by the aid of a table of logarithms find the distance OC.

[Logarithm tables required.]

XLIX.

1. What in plane geometry is the limit of the magnitude of an angle? In what sense in analytical trigonometry is the magnitude of an angle considered unlimited, and either positive or negative? Which of the ordinary trigonometrical functions are limited in magnitude, and which unlimited?

If $\sin\theta = a$, express by a general formula all the angles which have a for their sine; if $\cos A = b$, prove $a^2 + b^2 = 1$.

2. Find $\sin A$ and $\cos A$ in terms respectively of the sines and cosines of their supplements. Through what portions of the first four quadrants is the sine greater than the cosine in magnitude, and the tangent greater than the cotangent: when are they respectively equal and of the same sign?

Find sin 120°, tan 60°, cos 135°.

Prove $\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}}$.

3. Prove $\cos(A+B)=\cos A\cos B-\sin A\sin B$, and deduce from the formula the corresponding expressions for $\sin(A+B)$, $\sin(A-B)$, and $\cos 2A$.

4. Prove—(i.) $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\dfrac{A+B}{2}$; (ii.) $\cos 72^\circ=\dfrac{\sqrt{5}-1}{4}$;

* (iii.) $\sin A\sin(A+2C)+\sin B\sin(B+2A)+\sin C\sin(C+2B)=0$, if $A+B+C=180^\circ$.

5. Explain the notation $\sin^{-1}s$, $\tan^{-1}t$.

Prove—

(i.) $\sin^{-1}\dfrac{3}{5}+\sin^{-1}\dfrac{4}{5}=90^\circ$.

(ii.) $\tan^{-1}\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}+\tan^{-1}\sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}+\tan^{-1}\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}=90^\circ$,

where $s=\dfrac{a+b+c}{2}$.

6. Show that $\dfrac{\text{arc}}{\text{radius}}$ may be properly taken for the measure of an angle; of what angle is 1·5708 the circular measure?

If (θ) be the circular measure of an angle less than a right angle,

Prove $\sin\theta$ less than $\theta-\dfrac{1}{4}\theta^3$, and verify this when $\theta=\dfrac{\pi}{6}$.

7. In a plane triangle of which A, B, C are the angles, and (a), (b), (c) the sides subtending them,

Prove—

(i.) $c=a\cos B+b\cos A$.

(ii.) $\dfrac{a}{b}=\dfrac{\sin A}{\sin B}$.

* Deduce from these expressions the value $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$.

Prove $a^2+b^2+c^2=2ab\cos C+2ac\cos B+2bc\cos A$.

8. Find the area of a triangle in terms of any two sides and their included angle, and also in terms of the three sides. Find the area of the greatest triangle that has for two of its sides 50 and 60 feet.

9. Given the difference between the angles at the base of a triangle 17° 48′, and the sides subtending those angles 105·25 feet and 76·75 feet: find the angle included by the given sides.

10. An observer in a balloon when it is one mile high observes the angle of depression of a conspicuous object on the horizontal ground to be 35° 20′, then after ascending vertically and uniformly for 20 minutes, he observes the angle of depression of the same object to be 55° 40′: find the rate of ascent of the balloon in miles per hour.

11. A tower which stands on a horizontal plane is 200 feet high, and there is a small loophole in the tower at a certain height above the ground; an observer is at a horizontal distance from the tower of 300 feet, but stands on a mound so that his eye is 12 feet above the ground on which the tower stands, and in that position, the angles subtended at his eye by the portions of the tower above and below the loophole are equal: find the height of the loophole from the ground.

12. In a circle which has a radius of 10 feet two chords, AB, CD, are drawn at right angles to each other, and intersecting in O, AO, and CO are 3 and 4 feet respectively: find the sides and angles of the quadrilateral $ACBD$ formed by joining the extremities of the chords.

[Logarithm tables required.]

L.

1. In modern books on Trigonometry, what ratios are expressed by the sine, tangent, and secant of an angle respectively? Formerly the trigonometrical functions were defined as represented by straight lines: define the sine, tangent, and secant, according to this method, and show how the two systems are made consistent with each other. Define the complement and supplement of an angle, and prove $\cos A = \sin (90° - A) = -\cos (180° - A)$.

What is the formula corresponding to $\sin^2 A + \cos^2 A = 1$ if the linear definitions of the sine and cosine be adopted?

2. Trace the change in sign and magnitude of the secant of an angle from 0° to 360°; find cos 30°, tan 45°, and sec 60°. What are the three least positive angles that have $\frac{\sqrt{3}}{2}$ for their sine?

Given $\tan A = \cdot 5$, find $\sec A$.

Given $\cos 36° = \frac{\sqrt{5}+1}{4}$, find $\sin 18°$.

3. If $A+B$ be an angle less than a right angle, prove by a geometrical construction $\cos(A+B)=\cos A\cos B-\sin A\sin B$.

* Exhibit also the construction when A and B are each less than a right angle, and $A+B$ greater than a right angle.

Obtain the formula for $\sin(A-B)$ from the above value of $\cos(A+B)$.

4. Prove—

(i.) $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$.

(ii.) $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\dfrac{\tan\dfrac{A+B}{2}}{\tan\dfrac{A-B}{2}}$.

(iii.) $\tan 75^\circ=2+\sqrt{3}$.

(iv.) $\tan^{-1}\dfrac{1}{2}+\tan^{-1}\dfrac{1}{3}=45^\circ$.

* 5. Prove generally—

$$\cos\frac{A}{2}+\sin\frac{A}{2}=\pm\sqrt{1+\sin A},$$

$$\cos\frac{A}{2}-\sin\frac{A}{2}=\pm\sqrt{1-\sin A},$$

and express with proper signs the values of $\cos\dfrac{A}{2}$ and $\sin\dfrac{A}{2}$ when A lies between 270° and 360°.

From the above find sin 9° without reducing the surds.

6. In any plane triangle, if a, b, c are the sides subtending the angles A, B, C respectively, prove

(i.) $\sin\dfrac{A}{2}=\dfrac{1}{2}\sqrt{\dfrac{(a+b-c)(a+c-b)}{bc}}$.

(ii.) $\tan B=\dfrac{b\sin C}{a-b\cos C}$.

Verify formula (i.) if A be the greatest angle of a triangle whose sides are 30, 40, 50 feet.

7. Show that $\dfrac{\text{arc}}{\text{radius}}$ may be properly taken as the measure of an angle; find the angle which is the unit of measure when this mode of representing angles is referred to, and show also that the sine of a small angle is approximately equal to the circular measure of the angle itself.

If θ be the circular measure of an angle, show that the number of seconds it contains is equal to $\dfrac{\theta}{\sin 1''}$ nearly.

8. The sine of an angle defined as a ratio being less than unity, explain why the tabular logarithms of the sines of angles are expressed with whole numbers as characteristics. If the logarithm of the tangent of 18° be registered in the tables as 9·5117760, show what the tabular logarithm of cot 18° must be.

Find by the aid of the tables the value of A if $\cos A = \cot \alpha \tan \beta$ where $\alpha = 75° \ 30'$, $\beta = 28° \ 10'$.

9. In a triangle ABC, given $A = 35° \ 50'$, $BC = 595$ feet, $AC = 684$ feet, show that the solution will be ambiguous, and determine the third side for the greater of the two triangles to which the data belong.

10. From the top of a cliff on the sea-shore, 326 feet high, the angle of depression of a boat at sea was found to be 34° 30′: find the distance of the boat from the foot of the cliff.

11. An observer in a ship at sea sailing north observes two rocks A and B, to bear 25° east of his course; he then sails in a direction north-west for four miles, and observes A to bear due east and B north-east of his new position. Find the distance from A to B.

12. If π be the ratio of the circumference of a circle to its diameter, point out any method by which the approximate value of π may be determined numerically, and if r be the radius of a given circle show that the area of the circle is equal to πr^2.

13. Find the radius of the circle circumscribing a given triangle in terms of its sides.

If $ABCD$ be a quadrilateral that may be inscribed in a circle, AC a diagonal, show from the expression obtained above that the radius of the circle that would circumscribe the triangle ABC is equal to the radius of the circle that would circumscribe the triangle ACD, A and C being opposite angles of the quadrilateral.

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(*Obligatory.*)

LI.

1. What is meant by the circular measure of an angle?

If the moon be at a distance from the earth equal to sixty times the earth's radius, express—(1) in circular measure; (2) in minutes, the approximate value of the angle subtended at the moon by the radius of the earth.

2. Find the continued product of the following expressions :—

$$\sin(A-90^\circ)+\cos(A-180^\circ);\ \cot(A-90^\circ)-\tan(A-183^\circ);$$
$$\sec(A-93^\circ)-\operatorname{cosec}(A-180^\circ).$$

3. Find the value of $a\cos 2\theta+b\sin 2\theta$; when $\tan\theta=\frac{b}{a}$.

*** 4. Obtain the values of sec 225°; cot 435°; cosec 660°; sec 555°.**

5. Prove the following identities :—

(i.) $\sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$.

(ii.) $\frac{1+\sin A}{1+\cos A}=\frac{1}{2}\left(1+\tan\frac{A}{2}\right)^2$.

(iii.) $\cot^2\left(\frac{\pi}{4}+\frac{A}{2}\right)=\frac{2\operatorname{cosec}2A-\sec A}{2\operatorname{cosec}2A+\sec A}$.

*** 6. Express $\cos 5A$ in terms of powers of $\cos A$.**

7. In any plane triangle prove—

(i.) $\sin A:\sin B:\sin C::a:b:c$.

(ii.) $\frac{\sin A}{\sin B}=\frac{\cos A\cos C+\cos B}{\cos B\cos C+\cos A}$.

(iii.) the area $=\frac{a^2}{2}\sin B\sin C\operatorname{cosec}(B+C)$.

8. Prove that—

$$\sin^{-1}\sqrt{\frac{x}{a+x}}=\tan^{-1}\sqrt{\frac{x}{a}}=\frac{1}{2}\cos^{-1}\frac{a-x}{a+x}.$$

9. Show how to solve a triangle, having given two angles and a side.

In the triangle ABC find the side c, having given

$b = 100$ yards; $C = 62° 5'$; $A = 45° 41'$;
L sin 62° 5′ = 9·94627; log 9·2788 = ·96749;
L sin 72° 14′ = 9·97878.

10. Find the radii of the circumscribed and inscribed circles of a triangle in terms of its sides; and prove that the product of the radii, multiplied by the sum of the sines of the angles of the triangle, is equal to its area.

11. If from a point at which the elevation of the observatory on Ben Nevis is 60° a man walks 800 feet on a level plane towards the mountain, and then 800 feet further up a slope of 30° to a point at which he finds that the elevation of the observatory is 75°, show that the height of Ben Nevis is approximately 4478 feet, the man's path being always supposed to lie in a vertical plane passing through the observatory.

LII.

1. Find the relation between the arc of a given circle and the angle which it subtends at the centre. What is the angular unit when we say angle $= \frac{arc}{rad}$, and when $\pi = 180°$? On what understanding is the equation *angle = arc* true?

2. Define the secant of an angle, and trace its change in sign and magnitude through four right angles.

Prove that $\sin(\pi - A) = \sin A$ and $\tan(\pi + A) = \tan A$.

* 3. If the sine of an angle is given, show how to find the sine of half the angle, and explain why the general expression obtained ought to include four values. Given $\sin 300° = \frac{-\sqrt{3}}{2}$, find sin 150°.

4. Prove the following formulæ—

(i.) $$\sec A = \frac{1+\tan^2\dfrac{A}{2}}{1-\tan^2\dfrac{A}{2}}.$$

(ii.) $$\operatorname{cosec} A = \frac{1}{2}\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right).$$

(iii.) $$\frac{\cos 2B-\cos 2A}{\sin 2B+\sin 2A}=\frac{\sin 2A-\sin 2B}{\cos 2A+\cos 2B}=\tan(A-B).$$

5. Solve the equation—

$$\sin 6\theta - \sin\theta = \sin 2\tfrac{1}{2}\theta,$$

* and eliminate θ between the equations

$$x = a\cos\theta + b\cos 2\theta,$$
$$y = a\sin\theta + b\sin 2\theta.$$

6. Two sides and the angle between them being given, find the third side in a form adapted to logarithmic calculation.

7. Find the radii of the inscribed and circumscribed circles of a triangle.

Prove that the radius of the circle passing through the middle points of the sides of a triangle and the feet of the perpendiculars from the angular points is half that of the circumscribed circle.

8. From the top of a vertical tower, whose height is 100 feet, the angle of depression of an object is observed to be 60° 1′ and from the base to be 30° 1′; show that the vertical height of the base of the tower above the object is 50 feet approximately.

9. Find an expression for the area of a triangle in terms of the sides. If the triangle be right-angled, show that the expression reduces to half the product of the sides which contain the right angle.

10. One angle of a triangle is 32° 15′: the sides including another are 468 and 320 yards: find the latter angle.

Given $\log 2 = \cdot 3010300$ $\log 3 = \cdot 4771213$.
$\log 13 = 1\cdot 1139434$ $L \operatorname{cosec} 32° 15' = 10\cdot 2727724$.
$L \sin 21° 23' = 9\cdot 5621316$ $L \sin 51° 18' = 9\cdot 8923236$.

* 11. If θ be the circular measure of an angle less than a right angle, prove that $\sin\theta$ is greater than $\theta - \dfrac{\theta^3}{4}$.

What are meant by formulæ of verification?

LIII.

* 1. Explain what is meant by the circular measure of an angle. How many degrees are there in the angle whose circular measure is θ?

* 2. Prove from a figure that $\cos(A-B)=\cos A\cos B+\sin A\sin B$ when A lies between 315° and 360°, and $A-B$ between 180° and 225°.

* 3. Trace the changes in sign and magnitude of $\sin\theta-\sqrt{3}\cos\theta$, as θ increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$.

* 4. Find a formula expressing all the angles that have the same tangent as a given angle.

Find the value of sin 18°, and show from the value of $\cos 22\frac{1}{2}°$, that—

$$2\cos 5°\,37'\,30''=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}.$$

5. Prove the identities—

(i.) $\sec^2 A=1+\tan^2 A$.

(ii.) $\sin 3A=3\sin A-4\sin^3 A$.

(iii.) $\cos A+\cos(120°-A)+\cos(120°+A)=0$.

6. Solve the equations—

(i.) $2\sin\theta\sin 3\theta-\sin^2 2\theta=0$.

* (ii.) $\sin\frac{\theta}{2}=\operatorname{cosec}\theta-\cot\theta$.

(iii.) $$\left.\begin{aligned}\cos(x+y)+\cos(x-y)&=2\\ \sin\frac{x}{2}+\sin\frac{y}{2}&=0\end{aligned}\right\}.$$

* 7. Eliminate ϕ from the equations—

$$\left.\begin{aligned}p&=\sec\phi-\cos\phi\\ q&=\operatorname{cosec}\phi-\sin\phi\end{aligned}\right\}.$$

8. A, B, C are the angles of a triangle, a, b, c its sides, $2s$ the perimeter, and Δ its area; show that

(i.) $a\sin B=b\sin A$.

* (ii.) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$.

(iii.) $$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$
$$=\frac{2abc}{a+b+c}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

9. If R and r are the radii of the circles circumscribed about and inscribed in the triangle ABC; then—

$$* \text{(i.)}\ r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$\text{(ii.)}\ \Delta\,[\sin A+\sin B+\sin C]=\frac{r}{R}\,s^2.$$

10. A man ascends the slope of a hill inclined at an angle ω to the horizon; and twice during the ascent, at vertical heights a and b above the level of the plain, he turns round, and observes the angle of depression of the top of a tower standing in the plain directly in front of him. If α and β are these angles, find the height of the tower and its distance from the foot of the hill. [The side of the hill may be regarded as an inclined plane.]

11. The sides of a triangle, BC, CA, AB, are as 8 : 7 : 5. Find the angle A having given that

$L \tan 40° \; 54' = 9{\cdot}9376318$ $\log 2 = {\cdot}3010300$
$L \tan 40° \; 53' = 9{\cdot}9373765$ $\log 3 = {\cdot}4771213.$

12. The limit of $\frac{\sin\theta}{\theta}$ when θ is indefinitely diminished in unity.

LIV.

1. What are the characteristics of the logarithms of the numbers $(4261)^3$ and $({\cdot}0002)^2$? Given the logarithm of $25 = 1{\cdot}3979400$, find, without the tables, the logarithms of ·08 and 1·25.

2. Find by means of the tables, the value of $(372{\cdot}48) \times \left(\frac{150}{151}\right)^{60}$.

* 3. Define the sine and cosine of an angle; between what limits as to the magnitude of the angle will the sine and cosine both be negative? Without assuming the formula for $\cos(A+B)$, prove—

$$\cos 2A = (\cos A)^2 - (\sin A)^2.$$

4. Prove the following formulæ :—

(i.) $(\cos A + \sin A)^2 = 1 + \sin 2A$.

* (ii.) $\sin 18^\circ = \frac{\sqrt{5} - 1}{4}$.

* (iii.) $\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan A \tan C}$.

If A, B, C be the angles of a triangle, express from the above the relation between $\tan A$, $\tan B$, $\tan C$.

5. What is meant by the circular measure of an angle? What is the unit of angle referred to in the measure, $\text{angle} = \frac{\text{arc}}{\text{rad}^s}$? Find the circular measure of each angle of a regular pentagon.

6. Find the radius of the circle inscribed in a given triangle in terms of the sides of the triangle.

7. If a, b, c be the sides subtending the angles A, B, C of a plane triangle, find the value of $\tan\frac{A}{2}$ in terms of the sides, and in a form adapted to logarithmic computation. Apply the expression to find the angle A of the triangle in which $b = 37{\cdot}5$, $c = 50$, $a = 62{\cdot}5$.

8. For the solution of a triangle ABC, if $A = 52^\circ\ 19'$, $AC = 166{\cdot}5$, $BC = 162{\cdot}5$, show that there are two triangles to which the data belong; find also the third side of the greater of these two triangles.

9. An observer is 384 yards due south of a point in the horizontal plane from which a balloon ascended vertically, he measures from his position a base of 112 yards in a direction due east, and at the other extremity of this base finds the angle of elevation of the balloon to be $60^\circ\ 15'$: find the actual height of the balloon in yards at the instant of observation.

10. Two railroads intersect each other at an angle of $25^\circ\ 30'$; from the point of intersection a station on one line is distant 12 miles, and on the other line a station is distant 16 miles: find the distance of the stations from each other.

11. Find the side of a regular pentagon inscribed in the circle whose radius is ten feet.

LV.

1. Prove the formula for the sine of the difference of two angles, each of which is less than 90°. Show whether the formula will be altered by writing $(A-90°)$ and $(B-90°)$ in place of A and B.

* 2. Explain how angles are measured in Trigonometry. Find the number of grades, etc., in an angle of 57° 36′ 42″.

3. Prove the following formulæ—

(i.) $\cos 2A+\cos 2B=2\cos(A+B)\cos(A-B)$.

(ii.) $\sec 2A=\dfrac{\sec^2 A}{1-\tan^2 A}$.

(iii.) $\tan\left(45°+\dfrac{A}{2}\right)=\sqrt{\dfrac{1+\sin A}{1-\sin A}}$.

4. Are any triangles solved by Trigonometry for which solutions may not be found by Euclid? State the cases in which the solution of a triangle from three given parts is (1) impossible, (2) ambiguous.

5. In a triangle, if $\dfrac{\tan A}{\tan B}=\dfrac{\sin^2 A}{\sin^2 B}$, prove that the triangle is either isosceles or right-angled.

6. Two sides of a triangle are 76 and 57 yards respectively, and the angle opposite the former is 53° 7′ 49″; solve the triangle by the logarithms given below.

7. Prove the formula for the cosine of half an angle of a triangle in terms of the sides. The sides of a triangle being 400, 500, and 600 yards, calculate the angle opposite the side of 500 yards by the logarithms given below.

8. Two sides of a triangle being 180 and 140 yards, and the included angle 64° 12′, show how to find the other angles, and determine the number of degrees, minutes, and seconds in each of them.

9. A man in a balloon observes that two churches which he knows to be one mile apart subtend an angle of 11° 25′ 20″ when he is exactly over the middle point between them: find the height of the balloon.

10. Find an expression for R the radius of the circle described about a given triangle; and show that—

$$a\cos A+b\cos B+c\cos C=4R\sin A\sin B\sin C.$$

Some of the following logarithms will be required.

$\text{Log}\,5{\cdot}7={\cdot}75587$
$\text{Log}\,7{\cdot}6={\cdot}88081$
$\text{Log}\,9{\cdot}5={\cdot}97772$

$\text{L}\sin 38^\circ\,18'=9{\cdot}79224$
$\text{L}\sin 27^\circ\,32'=9{\cdot}66489$
$\text{Log}\,2={\cdot}30103$
$\text{Log}\,3={\cdot}47712$
$\text{Log}\,5={\cdot}69897$

$\text{L}\cot 20^\circ=10{\cdot}43893$
$\text{L}\cot 32^\circ\,6'=10{\cdot}20253$

$\text{L}\sin 53^\circ\,7'=9{\cdot}90301$ diff. for $1'=10$
$\text{L}\sin 36^\circ\,52'=9{\cdot}77812$ diff. for $1'=17$
$\text{L}\cos 41^\circ\,45'=9{\cdot}87277$ diff. for $1'=24$
$\text{L}\cos 10^\circ=9{\cdot}99335$
$\text{L}\cos 27^\circ\,53'=9{\cdot}94640$ diff. for $1'=6$
$\text{L}\tan 11^\circ\,16'=9{\cdot}29932$ diff. for $1'=66$
$\text{L}\tan 19^\circ\,43'=9{\cdot}55434$ diff. for $1'=40$

$\text{L}\cos 18^\circ\,26'=9{\cdot}97713$ diff. for $1'=6$
$\text{L}\tan\ 5^\circ\,42'=8{\cdot}99919$ diff. for $1'=127$.

LVI.

1. Define the tangent of an angle. Prove that $\tan(180^\circ+A)\cot A=1$. Show whether there are any angles for which the tangent is equal to the cosine or secant or cosecant, respectively.

2. Prove that $\sin(A+B)=\sin A\cos B+\cos A\sin B$. Deduce from this the corresponding expression for $\cos(A+B)$; and find a value for $\cos 75^\circ$.

3. Prove the following formulæ:—

(i.) $$\sin 2A=\frac{2\tan A}{1+(\tan A)^2}.$$

(ii.) $$\tan A\tan B=\frac{\left(\cos\dfrac{A-B}{2}\right)^2-\left(\cos\dfrac{A+B}{2}\right)^2}{\left(\cos\dfrac{A-B}{2}\right)^2-\left(\sin\dfrac{A+B}{2}\right)^2}.$$

Show that this is true when $A=120^\circ$, $B=60^\circ$.

4. Find an expression for $\sin A$ in terms of $\sin 2A$. Show by a diagram (or otherwise) which are the four angles to which the four values of this expression severally belong when $2A = 45°$.

5. Find the circular measure of 30°; and the minutes and seconds in the angle which a distance of 50 miles on the earth's surface subtends at the centre, taking the earth's radius at 4000 miles.

6. In any plane triangle prove—

$$\text{(i.)}\ \frac{\sin A}{\sin B} = \frac{a}{b}.$$

$$\text{(ii.)}\ \frac{b^2 - c^2}{a} = b \cos C - c \cos B.$$

7. Given the base of an isosceles triangle, and the perpendicular from the vertex on the base; show how to solve the triangle.

Ex. The base of an isosceles triangle and the perpendicular upon it from the vertex are each 352·72 yards, find the sides and angles.

8. Find an expression for the area of a triangle in terms of the sides. Deduce the area of a regular hexagon with a perimeter of 30 feet.

9. The logarithms of trigonometrical ratios as given in the ordinary tables are not precisely the logarithms of the numerical values of those ratios: how do they differ from them, and what is the reason for so tabulating them? For which of the ratios is the difference for successive angles *subtractive*?

Given $\log a$ and $\log b$, and $\tan \theta = a + b$, to find $L \tan \theta$; show that the process may be shortened by the use of a subsidiary angle.

log 2 = ·3010300
log 3·5273 = ·5474424
log 3·9436 = ·5958929, diff. 110
L tan 26° 33′ 50″ = 9·6989480, diff. for 10″ 526
L tan 63° 26′ = 10·3009994 ,, ,,
L cos 26° 34′ = 9·9515389 ,, 105
L cos 63° 26′ 10″ = 9·6504974 ,, 421.

LVII.

1. Trace the changes in sign and magnitude of the secant of an angle as the angle varies from 0° to 360°. If you had to find sec 127° by the ordinary tables which give the secants of angles from 0° to 90° only, how would you do it?

2. Prove the formulæ—

(i.) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

(ii.) $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4\cos^2\dfrac{A-B}{2}$.

(iii.) In any triangle $\tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{A}{2}\tan\dfrac{C}{2} = 1$.

* 3. Prove the formula $2\sin A = \pm\sqrt{1+\sin 2A} \pm \sqrt{1-\sin 2A}$. Explain the four values, and show how in any given case to choose the right one.

4. Define the circular measure of an angle. If θ is the circular measure of an angle, which is one-third of a right angle, compute the value of $\dfrac{\sin\theta}{\theta}$. $\left(\text{Take } \dfrac{1}{\pi} = {\cdot}3183.\right)$

5. Two angles of a triangle are 15° and 45°, and the included side 20 feet; show that the area of the triangle is $100 - \dfrac{100}{\sqrt{3}}$.

6. Two sides of a triangle are 23031 and 7677 yards respectively, and the included angle 30° 10′ 5″; find the other angles.

log 1·5354 = ·1862215
log 3·0708 = ·4872515
L tan 15° 5′ = 9·4305727
diff. for 10″, 838
L cot 61° 40′ 20″ = 9·7316452
diff. for 10″, 504.

7. When a triangle ABC is solved from the given parts $AB = 100$, $BC = 56$, the angle $BAC = 30°$, find the difference of the two values obtained for the angle B: having given—

log 1·12 = ·049218
L cos 26° 45′ = 9·950838
L cos 26° 46′ = 9·950778.

8. Express the radius of the circle inscribed in a given triangle in terms of the sides of the triangle. If the sides of the triangle are 51, 68, and 85 feet, show that the shortest side is divided by the point of contact of the inscribed circle into two segments, one of which is double of the other.

9. Find the area of a regular hexagon. Also if its alternate angles be joined so as to form another regular hexagon, and again the alternate angles of the latter hexagon be joined, and so on, show that the sum of the areas of all the figures so formed $=\frac{A}{2}$, where A is the area of the original hexagon.

* 10. Show that in a table of cosines we may expect to find the "differences" least at the beginning, and increasing as the table proceeds. Also prove that the change of the tabular *logarithmic* sine of an angle is approximately proportional to the change of the angle.

* 11. Write down the exponential values of $\sin\theta$ and $\cos\theta$, and apply them to show that $\cot\frac{\theta}{2}=\frac{\sin\theta}{1-\cos\theta}$.

LVIII.

1. Define the sine of an angle, and state the convention by which its algebraic sign is determined. Find all the values of θ less than two right angles which make $\sin 3\theta+\cos 3\theta=0$.

2. In the ordinary tables the sines and cosines of angles are recorded up to 45° only: explain how those of angles up to 180° may thence be found. Show why an angle which is nearly a right angle cannot be found so accurately when its sine is given as when its cosine is given.

3. Find by logarithms $\frac{(\cdot 006)^5}{125}$. Also find L sin 60°.

4. Prove the following formulæ—

$$(\alpha)\ \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

$$(\beta)\ \sin 2A=\frac{1-[\tan(45^\circ-A)]^2}{1+[\tan(45^\circ-A)]^2}.$$

$$(\gamma)\ \sin 3A=4\sin A\cos(30^\circ+A)\cos(30^\circ-A).$$

5. Define the circular measure of an angle. Apply it to find the distance at which a building 50 feet wide will subtend an angle of 3′.

6. Prove that $\theta = \pm 30^\circ$ satisfies the equation—

$$\tan(45^\circ - \theta) + \cot(45^\circ - \theta) = 4.$$

7. In a triangle, of which a, A, and c are given, show that—

$$a \cos C = \pm \sqrt{a^2 - c^2 (\sin A)^2}\,;$$

and that if $\tan\phi = \dfrac{a}{c \cos A}$

$$\cos C = \frac{\pm \sqrt{\sin(\phi - A)\sin(\phi + A)}}{\sin\phi \cos A}\,;$$

also that ϕ will always be nearer to a right angle than A.

8. A man observes the elevation of a mountain top to be 15°, and after walking 3 miles directly towards it on level ground the elevation is 18°; find his distance from the mountain.

9. Investigate the area of a triangle of which the sides are given. Find the area of a triangle of which the sides are 5, 7, and 8 chains respectively.

10. In a right-angled triangle prove $\cot\dfrac{A}{2} = \dfrac{b+c}{a}$, if C be the right angle. If the logarithms of a, b, and c be given, show how that of $\cot\dfrac{A}{2}$ may be found by help of the tables without first finding b and c.

Ex. $\log a = \cdot7525750$, $\log b = \cdot30103$, $\log c = \cdot7781513$.

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log 1·4109 = ·1494962, diff. 308
log 1·7109 = ·2332246, diff. 254
log 1·7320 = ·2385479, diff. 251
log 2 = ·3010300
log 3 = ·4771213
log 3·14159 = ·4971495
log 5·7295 = ·7581167 diff. 76
log 6·2208 = ·7938465

L sin 3° = 8·7188002
L sin 15° = 9·4129962
L cos 15° = 9·9849438
L sin 18° = 9·4899824
L cos 18° = 9·9782063
L tan 60° = 10·2385606
L cos 60° = 9·6989700

L cot 35° 15′ 50″ = 10·1505231; diff. for 10″, 447.

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(Obligatory.)

LIX.

1. Explain the English and French methods of measuring angles, and show how, when the measure of an angle according to either method is known, its measure according to the other may be found.

What would be the French measure of an angle containing 100 degrees?

2. Define the *tangent* of an angle, and trace its changes in sign and magnitude as the angle increases from 0 to 360°.

Show that $\tan A = \tan (180° + A)$.

3. Investigate a formula including all angles which have the same cosine.

Find all the values of θ which satisfy the equation—

$$\tan \theta - 2 \sin \theta = 0.$$

4. Prove that—

$$\sin (A + B) = \sin A \cos B + \sin B \cos A.$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$

If $x = c \tan \theta$, show that—

$$(x^2 + c^2)^2 \sin 4\theta = 4cx (c^2 - x^2).$$

5. Establish the formulæ—

(i.) $\cot A - \tan A = 2 \cot 2A.$

(ii.) $\tan A + \sec A = \cot \left(45 - \frac{A}{2}\right).$

(iii.) $(\tan A - \tan B)^2 = \sec^2 A + \sec^2 B - 2 \sec A \sec B \cos (A - B).$

* 6. Show how to find $\cos A$ when $\cos 2A$ is known.

Explain the ambiguity in the result, and show that it is just such as might have been expected to present itself.

Find the cosine of 112° 30′.

* 7. f θ be the circular measure of an angle less than a right angle, show that $\sin\theta$ is less than θ, but greater than $\theta - \frac{\theta^3}{4}$.

Hence show how the sine of 10″ may be calculated approximately.

8. Prove that the sides of any plane triangle are proportional to the sines of the opposite angles.

The side BC of the triangle ABC is bisected in D and produced to E, show that the cotangents of the angles CAD, BAC, ACE are in arithmetical progression.

9. Show that in any plane triangle—

$$\text{(i.)}\ \cos^2\frac{C}{2} = \frac{(a+b+c)\,(a+b-c)}{4ab}.$$

$$\text{(ii.)}\ \tan\frac{1}{2}\,(A-B) = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

The lengths of two sides of a triangle, including an angle of 60°, are $3\sqrt{3}+2$, and $3\sqrt{3}-2$ respectively; find the other angles and the third side, having given—

$$\log 2 = \cdot 301030,\ \log 3 = \cdot 47712$$
$$\text{L}\tan 33^\circ\ 41' = 9\cdot 82380$$
$$\text{L}\tan 33^\circ\ 42' = 9\cdot 82407.$$

10. Find the area of regular polygon of n sides, and show that the sum of the radii of its inscribed and circumscribed circles is $\frac{a}{2}\cot\frac{\pi}{2n}$, where a is the length of one of its sides.

* 11. A man standing due south of a spire finds the angular elevation of its summit to be α. He then walks to a point a yards due west of his former position and finds the elevation to be B. Show that the height of the spire in yards is $\frac{a\sin\alpha\sin\beta}{\sqrt{\sin(\alpha-\beta)\sin(\alpha+\beta)}}$.

12. If the triangle ABC be right-angled at C, show that

$$2\tan^{-1}\frac{a}{b+c} = \cos^{-1}\frac{b}{c}.$$

Questions set in the higher mathematical papers.

1. If R, r, r_1, r_2, r_3, are the radii of the circumscribed, inscribed, and escribed circles of a triangle, prove that $r_1 + r_2 + r_3 - r = 4R$.

2. Sum the series—

* (i.) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots$

(ii.) $\sin\theta \sec 3\theta + \sin 3\theta \sec 9\theta + \sin 9\theta \sec 27\theta + \ldots$

each to n terms: * and prove that—

$$\tan\theta + \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + \ldots = \frac{1}{2}\log\frac{\cos\left(\theta - \frac{\pi}{4}\right)}{\cos\left(\theta + \frac{\pi}{4}\right)}.$$

* 3. If n be a positive integer, expand $(\cos\theta)^{2n+1}$ is a series of cosines of multiples of θ.

Express $\cos^5\theta\sin^2\theta$ in terms of cosines of multiples of θ.

LX.

1. Define the circular measure of an angle, and show how to find the circular measure of an angle of n degrees.

Find the number of degrees in the unit of circular measure to three places of decimals, assuming that the circumference of a circle bears to its diameter the ratio 22 : 7.

2. Show that $\tan(90 + A) = -\cot A$, and that $\tan(180° + A) = \tan A$. Find $\tan 60°$. * Deduce the values of $\cot 240°$ and $\tan 300°$.

3. Trace the changes of $\cos A + \sec A$ as A changes from 45° to 135°.

4. Find the value of $\tan(A + B)$ in terms of $\tan A$ and $\tan B$, and deduce the value of $\tan 3A$ in terms of $\tan A$.

Show that $\tan 75° - \tan 60° = 2\tan 45°$.

5. Establish the formulæ—

(i.) $\sec^2 A - \cos^2 A = \sin^2 A + \tan^2 A$.

(ii.) $\dfrac{\sin A + \sin 2A}{\cos A - \cos 2A} = \cot\dfrac{A}{2}$.

(iii.) $\sin^2 2A = \dfrac{4\cos 2A}{\cot^2 A - \tan^2 A}$.

6. If $A + B + C = 90°$, show that—

$$\sin 2A + \sin 2B + \sin 2C = 4\cos A\cos B\cos C,$$

* and if $A + B + C = 180°$, show that—

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C.$$

7. If $a\sin\theta + b\sin(\theta + \alpha) = c$, and $a\cos\theta + b\cos(\theta + \alpha) = d$, find an expression for $\cos\alpha$ in terms of a, b, c, d.

* 8. Obtain formulæ for the solution of a triangle when two sides and the angle opposite to one of them are given.

Explain why two solutions sometimes occur.

9. If a, b, c be the sides of a triangle, A, B, C, the angles opposite to them respectively, show that—

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}}$$

where $s=\frac{a+b+c}{2}$.

If p be the altitude of the triangle, and a the base, show that—

$$\frac{p}{a}=\frac{\sin B\sin C}{\sin A}.$$

10. Find expressions for the radii of the circles inscribed in and circumscribed about a triangle.

* The sides of a triangle are as $2:3:4$; show that the radii of the escribed circles are as $\frac{1}{5}:\frac{1}{3}:1$.

11. On the same base, and on the same side of it, a series of isosceles triangles are described, their vertical angles being A, $2A$, $4A$, etc.; show that the distances between their vertices are proportional to cosec A, cosec $2A$, cosec $4A$, etc.

12. The distance between two boats, each moored in the middle of a river, in a straight reach of uniform breadth, subtends an angle of 60° at the eye of an observer standing on the bank of the river opposite to one of the boats. He walks 100 yards away from the river, and the distance between the boats then subtends an angle of 30°. Find the breadth of the river.

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1. Assuming De Moivre's Theorem to be true for positive integers, prove it to be so for negative and fractional indices.

Find expressions for the three values of $\{1-\sqrt{-1}\}^{\frac{1}{3}}$.

2. Prove that—

$$\text{(i.)}\quad \frac{\sin x}{x}=\cos\frac{x}{2}\,\cos\frac{x}{4}\,\cos\frac{x}{8}\ldots$$

$$\text{* (ii.)}\quad \log\frac{a}{c}=\frac{b}{a}\cos C+\frac{b^2}{2a^2}\cos 2C+\frac{b^3}{3a^3}\cos 3C+\ldots$$

in a plane triangle, a being supposed greater than b.

LXI.

1. Prove that the circumferences of circles vary as their radii. Explain the advantages of measuring angles by the English, French, and circular methods respectively.

The angles of a triangle are in geometrical progression, and the number of grades in one is to the circular measure of another as $50 : \pi$; find the two solutions.

2. Find the other five trigonometrical ratios of an angle in terms of its cotangent.

Prove that if two angles have the same sine, and also any of the other five trigonometrical ratios (with one exception) the same, they will differ by a multiple of 2π.

* 3. Show how to express $a \sin\theta + b \cos\theta$ in the form $c \sin\phi$ where c is a constant, and hence trace the changes in—

$$\sin\theta + \sqrt{3} \cos\theta$$

as θ varies from 0 to 2π.

4. Prove the formula for $\cos(A - B)$ taking a figure in which

$$A = 260°, B = 120°.$$

If $\sin(A - B) = \sin A + \sin B$, prove that either A or B or $A - B$ must be an exact number of right angles.

5. Eliminate θ and ϕ from—

$$\sin^2\theta \cos\gamma = \cos\alpha,\ \sin^2\phi \cos\gamma = \cos\beta, \text{ and } \tan\theta \tan\alpha = \tan\phi \tan\beta.$$

* 6. If $A + B + C = \pi$, prove that—

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C,$$

and that—

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2} \cot\frac{B}{2} \cot\frac{C}{2}.$$

Express $\sin 3A$ and $\cos 3A$ in terms of $\sin A$ and $\cos A$.

* 7. Show how to solve a triangle having given two sides and the included angle.

Change the formula $a^2 = b^2 + c^2 - 2bc \cos A$ into one adapted for logarithmic computation.

8. Write down the formula for the sine and cosine of any angle of a triangle in terms of the sides. Find the area of a triangle in terms of two sides and the included angle. Hence deduce an expression for the area in terms of the sides only.

9. The sides of a quadrilateral taken in order are 4, 5, 8, 9, and one diagonal is 9; find its angles and area.

* 10. Find the distance between two inaccessible objects on a horizontal plane.

Find by the measurement of straight lines only, the position of an inaccessible object on a horizontal plane.

11. Prove that the sum of the radii of the escribed circles of a triangle is equal to the radius of the inscribed circle together with twice the diameter of the circumscribed circle.

Construct geometrically, *or* solve trigonometrically, a triangle in which the base, the radius of the inscribed circle, and the radius of the circle escribed on the base, are given.

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1. Establish the exponential expressions for $\sin \theta$ and $\cos \theta$.

If $C + Si = e^{x \cdot e^{i\theta}}$, where $i = \sqrt{-1}$, find the rational values of C and S.

2. Prove that—

* (i.) $\sin \theta = \theta \left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right) \cdots$

(ii.) $\frac{1}{\sqrt{-1}} \log \frac{x - \sqrt{-1}}{x + \sqrt{-1}} = \tan^{-1} x.$

LXII.

1. Point out any advantages in Trigonometry that result (1) from considering the magnitude of an angle unlimited, (2) from considering that angles and lines may have negative as well as positive values. Define those of the elementary trigonometrical functions that may become infinite in value, and trace the signs and magnitudes of one of these as the angle varies from 0° to 360°.

Prove $\cos A = \cos(-A)$. $\quad$ $\operatorname{Sin} A = -\sin(-A)$.

2. Find $\tan 60°$, and prove that $(\sin 30°)^2$, $(\sin 45°)^2$, $(\sin 60°)^2$ are as the numbers 1, 2, 3. * Express the values of $\sin 660°$, $\tan 660°$, $\operatorname{vers} 660°$. Trace the changes in the sign and value of $(\cos A - \sin A)$ as (A) varies from 0° to 360°.

* 3. Prove, by means of a geometrical figure, $\cos 2A = (\cos A)^2 - (\sin A)^2$, without assuming the formula for $\cos(A+B)$, and deduce from the above value of $\cos 2A$ that of $\sin 2A$.

4. Find $\sin 18°$, and thence determine $\sin 81°$ without reducing the surds that may be involved in the expression.

If $\cos 2A = \frac{\sqrt{5}+1}{4}$, prove one value of A to be 18°.

5. Prove—

$$\text{(i.)} \quad \frac{\cos A + \tan\frac{A}{2}\sin A}{\sin A - \tan\frac{A}{2}\cos A} = \cot\frac{A}{2}.$$

$$\text{(ii.)} \quad \tan 3A = \frac{3\tan A\,(\sec A)^2 - 4(\tan A)^3}{4 - 3(\sec A)^2}.$$

If $\cos A - \sin A = \frac{1}{\sqrt{5}}$, find $\tan A$, and thence two values of A.

6. If A, B, C are the angles of a plane triangle, a, b, c the sides subtending them, and s half the perimeter,

Prove—

$$\text{(i.)} \quad \cos\frac{A}{2} = \sqrt{\frac{s\,.\,s-a}{bc}}.$$

$$\text{(ii.)} \quad c^2 = (a-b)^2\left(\cos\frac{C}{2}\right)^2 + (a+b)^2\left(\sin\frac{C}{2}\right)^2.$$

* Show how to adapt (ii.) to logarithmic computation by introducing a subsidiary angle.

7. Find an equation that expresses the relation between the circular measure of an angle and the measure of the same angle in degrees.

Find the circular measure of an angle of a regular quindecagon. If θ is expressed in circular measure prove

$$\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\ .\ .\ .\ \cos\frac{\theta}{2^n}=\frac{\sin\theta}{\theta}$$

when (n) is indefinitely increased.

8. If (A) be the area of a triangle whose three sides are a, b, c, prove that the radius of the circle circumscribed about the triangle is $\frac{abc}{4A}$.

In the ambiguous case for the solution of a triangle show that the radii of the circles circumscribing each of the triangles to which the data apply are equal to each other.

9. The three sides of a triangle are 320, 440, 560 feet respectively; find its greatest angle *or* its area.

10. From the intersection of two straight paths which are inclined at an angle of 37°, two pedestrians, A and B, start at the same instant to walk along the paths, A at the rate of 5 miles an hour, and B at a uniform rate also; after 3 hours they are found to be $9\frac{1}{2}$ miles apart; show that there are two rates at which B may walk to fulfil the condition, and find either of those rates.

11. AB and DE are two chords of a circle at right angles to each other intersecting in C, $AC=40$ feet, $DC=30$ feet, and the radius of the circle is 100 feet. Find the sides of the quadrilateral $ADBE$, and determine two of its opposite angles.

---

* 1. Prove De Moivre's Theorem for a positive integer, and hence prove that—

$$\cos\frac{m}{n}\theta+\sqrt{-1}\sin\frac{m}{n}\theta$$

is one of the values of $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{m}{n}}$.

What are the other values?

2. Prove that—

$$\cos\theta + \sqrt{-1}\sin\theta = e^{\theta\sqrt{-1}}.$$

Hence prove that—

$$\sin(\theta+\phi\sqrt{-1}) = \sin\theta.\frac{e^{\phi}+e^{-\phi}}{2} + \sqrt{-1}\cos\theta.\frac{e^{\phi}-e^{-\phi}}{2}.$$

* 3. Prove that—

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \text{etc.}$$

Prove that—

$$\left(3^{\frac{1}{2}} - 3^{-\frac{1}{2}}\right) - \frac{1}{3}(3^{\frac{3}{2}} - 3^{-\frac{3}{2}}) + \frac{1}{5}(3^{\frac{5}{2}} - 3^{-\frac{5}{2}}) - \text{etc.} = \frac{\pi}{6}.$$

LXIII.

1. Give the ratio definitions of the elementary trigonometrical functions, sine, cosine, tangent, secant. These functions have also been defined as straight lines; give these definitions, and show from them that tan 90°, sec 90°, would each be infinite. Show that the two systems are consistent. How would the tabulated logarithms, such as $\sin 25° = 9{\cdot}6250483$ be interpreted, in each system respectively?

2. Prove $\sin(90°+A)=\cos A$, $\cos(90+A) = -\sin A$. If one angle of a right-angled triangle is 30°, show that the side opposite to it is half the hypothenuse, hence find sin 60°; write down, with their proper signs, the values of sin 570°, tan 630°. What are the three least angles to satisfy the equation $\cos\theta = -\frac{1}{2}$?

3. Prove $\sin(A-B) = \sin A \cos B - \cos A \sin B$, drawing the figure for A less than 90° and B less than A. Verify the equation when $A=60°$ and $B=30°$. If in the above expression for $\sin(A-B)$, $A-90°$ be substituted for A and $90°-B$ be substituted for B, what does the equation become?

4. Prove (1) $\tan(45°+A)+\tan(45°-A) = 2\sec 2A$.

* (2) $$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\sin\frac{A+B}{4}\sin\frac{A+C}{4}\sin\frac{B+C}{4}$$

if $A+B+C=180°$.

5. Given $\sin 54^\circ = \frac{\sqrt{5}+1}{4}$ find $\sin 27^\circ$ in a surd form.

Prove $\frac{\pi}{4} = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$.

What particular use has been made of this value of $\frac{\pi}{4}$?

6. Show that the angle subtended by an arc equal in length to the radius is invariable for every circle. Calculate the value of this angle in degrees. Express 22° 30′ in circular measure. Show that $\frac{\sin\theta}{\theta} = 1$, as θ is indefinitely diminished and expressed in circular measure. Is this true if the angle be expressed in degrees?

7. In any plane triangle of which a, b, c are the sides subtending the angles A, B, C respectively,

Prove—

(1) $\frac{\operatorname{cosec} A}{\operatorname{cosec} B} = \frac{b}{a}$;

(2) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$;

(3) $\cos\frac{A-B}{2} = \frac{(a+b)\sin\theta}{2\sqrt{ab}}$, if $\cos\theta = \frac{a-b}{c}$.

8. Given the three sides of a triangle, find the radius of the escribed circle which touches one of those sides and the other two sides produced.

If the area of a triangle be 96 square feet, and the radii of the three escribed circles be respectively 8, 12, 24 feet, find the sides of the triangle.

9. In any triangle, when the perpendicular upon the base from the opposite angle falls within the base, prove that the difference of the segments of the base is to the difference of the other two sides of the triangle as the sum of those sides is to the base.

By means of this property find the least angle of the triangle whose sides are 50, 60, 88 feet.

10. If the solution of a triangle were required from the following data respectively—

(1) $A = 43° 15'$, $AB = 36{\cdot}5$ feet, $BC = 20$ feet,
(2) $A = 43° 15'$, $AB = 36{\cdot}5$ feet, $BC = 30$ feet,
(3) $A = 43° 15'$, $AB = 36{\cdot}5$ feet, $BC = 45$ feet,

point out which solution would be impossible, and which would be ambiguous; find the third side for the triangle, the solution of which is neither impossible nor ambiguous.

11. Two observers at stations on the sea-shore are half a mile apart. A perpendicular from a rock at sea falls on the base line between the stations, and the horizontal angles which the rock makes with the base at its extremities are observed to be 65° 20′, 70° 30′ respectively; find the least distance of the rock from the base.

12. A chord OB is drawn through a point O in the circumference of a circle whose radius is 10 feet, making an angle of 36° with the tangent to the circle at the point O; show that OB is the side of a regular pentagon inscribed in the circle, and find its length and the area of the pentagon.

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1. Prove—

$(\cos a_1 \pm \sqrt{-1} \sin a_1)(\cos a_2 \pm \sqrt{-1} \sin a_2)(\cos a_3 \pm \sqrt{-1} \sin a_3)$, etc. to n factors

$$= \cos(a_1 + a_2 + a_3 + \text{etc.}) \pm \sqrt{-1} \sin(a_1 + a_2 + a_3 + \text{etc.}),$$

and hence deduce De Moivre's Theorem when the index is a whole number.

2. Assuming the expansion of $(\cos A + \sqrt{-1} \sin A)^3$, obtain from the result the known trigonometrical formulæ for $\cos 3A$ and $\sin 3A$ in terms of $\cos A$ and $\sin A$ respectively.

3. Obtain the following series—

$$\cos a = 1 - \frac{a^2}{1.2} + \frac{a^4}{1.2.3.4} - \text{etc.},$$

and write down the corresponding series for $\sin a$. In what measure is the angle a here expressed? Show that the series for $\cos a$ is convergent.

LXIV.

1. Define the principal trigonometrical ratios, and find a formula for all the angles having the same sine. Find all the positive solutions less than 2π of the equation $\sin 3\theta = \cos 2\theta$.

2. Prove by independent geometrical construction in each case—

$$\text{(i.)}\quad \cos(A+B) = \cos A \cos B - \sin A \sin B,$$

$$*\ \text{(ii.)}\quad \cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2};$$

and find sec 15° and cos 36°.

3. Given that two sides and the included angle of a triangle are 7, 3, and 70° respectively, find the logarithm of the third side, having given that—

$L \cot 35° = 10·154773$ $\qquad$ $L \tan 29° 44' 15'' = 9·756833$
$L \sin 35° = 9·758591$ $\qquad$ $L \cos 29° 44' 15'' = 9·938673$
$\log 2 = ·3010300.$

* 4. The sides of a triangle are as $4 : 5 : 6$; find the angle opposite to the side 5. Given that

$L \cos 27° 53' = 9·9464040$ $\qquad$ $L \cos 27° 54' = 9·9463371$ $\qquad$ $\log 2 = ·3010300.$

5. In the triangle ABC, given $A = 41°$, $a = 141$, $b = 172·5$. Find B and C. Given

$L \sin 41° = 9·816943$ $\qquad$ $L \sin 53° 22' = 9·904429$
$\log 172·5 = 2·236789$ $\qquad$ $L \sin 53° 23' = 9·904523$
$\log 141 = 2·149219.$

6. Find the radius of the circle touching one side of a triangle and the two other sides produced. Find also the length of the chord joining the points of contact.

7. Prove De Moivre's Theorem for all forms of the index, and prove that the expression $\sqrt[n]{a + b\sqrt{-1}} + \sqrt[n]{a - b\sqrt{-1}}$ has n real values.

$$\text{Example: } \sqrt[3]{1+\sqrt{-3}} + \sqrt[3]{1-\sqrt{-3}}.$$

* 8. Find exponential values for $\cos\theta$ and $\sin\theta$; and prove that the coefficient of θ^{2n} in $e^{\theta}\cos\theta$ is $\dfrac{2^n}{\underline{|2n}}\cos\dfrac{n\pi}{2}$.

* 9. Express $\sin^n\theta$ in terms of sines of multiples of θ, where n is of the form $4m - 1$.

LXV.

1. Define the sine, tangent, and versed sine of an angle. For what values of A will $\sin A$ be less than $\operatorname{versin} A$?

Prove that for all values of A, $\cot A + \tan A = \operatorname{cosec} A \sec A$.

2. Prove the following formulæ :—

$$\text{(i.)} \quad \sin(A+B) = \sin A \cos B + \cos A \sin B.$$

$$* \text{ (ii.)} \quad 2 \operatorname{cosec} 4A + 2 \cot 4A = \cot A - \tan A.$$

$$\text{(iii.)} \quad \cos 18^\circ \cos 54^\circ = \frac{\sqrt{5}}{4}.$$

The value of $\sin 18^\circ$ in (iii.) not being assumed.

3. If a, b, and c be the sides of the triangle ABC, prove that

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Given $b=2$, $c=3$, and $A = \cos^{-1}\frac{1}{3}$; find a, $\cos B$, and $\cos C$.

4. In the triangle ABC, given $A=42^\circ$, $b=1706$, $a=1401$; find B and C.

Given $\mathrm{L} \sin 42^\circ = 9{\cdot}8255109$ $\qquad$ $\mathrm{L} \sin 54^\circ\, 34' = 9{\cdot}9110460$
$\log 1706 = 3{\cdot}2319790$ $\qquad$ $\mathrm{L} \sin 54^\circ\, 35' = 9{\cdot}9111359$.
$\log 1401 = 3{\cdot}1464381$

5. Given that the quadrilateral, whose sides are a, b, c, and d, can be inscribed in a circle, find its area.

Prove that the radius of the circumscribing circle is equal to

$$\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{4 \text{ area}}}$$

* 6. Prove De Moivre's Theorem for all forms of the index.

7. Write down the expansions of $\sin\theta$ and $\cos\theta$ in powers of θ, and deduce the exponential expressions of $\sin\theta$ and $\cos\theta$.

* Find x and y from the equation—

$$x + y\sqrt{-1} = \log(A + B\sqrt{-1})$$

* 8. Express $x^n - 1$ in quadratic factors, n being an even integer.

LXVI.

1. Define the tangent of an angle, and trace the variations in its sign and magnitude as the angle increases from zero to four right angles. Trace the changes in sign of the expression $\tan 3\theta . \tan 4\theta$, as θ changes from 0 to π.

* 2. Prove by means of geometrical figures the formulæ—

$$\text{(i.)} \cos(A-B)=\cos A \cos B+\sin A \sin B.$$

$$\text{(ii.)} \tan(A+B)=\frac{\tan A \tan B}{1-\tan A \tan B}.$$

* 3. Find an expression for all the angles which have a given tangent.

Having given $\tan A$, find $\tan \frac{A}{2}$, and explain, by means of a figure, why two values are obtained.

Prove that $\tan 37° \, 30' = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$.

4. Find expressions, in terms of the sides of a triangle, for the radii, r and R, of the inscribed and circumscribed circles. Prove that the area of the triangle is equal to $Rr(\sin A+\sin B+\sin C)$. Prove also that the distance between the centres of the inscribed and circumscribed circles is $\sqrt{R^2-2Rr}$.

5. Having given two sides and the included angle of a triangle, show how to find the remaining parts by help of a table of logarithms.

At the top C of a tower, of height h, the angles of depression of two objects A, B, on the horizontal plane on which the tower stands, are $\frac{\pi}{4}-a$ and $\frac{\pi}{4}+a$ respectively, and the angle subtended by the objects is $2a$; prove that $AB=2h \tan 2a$, and that the angles CAB, CBA are $\frac{\pi}{4}-a$ and $\frac{3\pi}{4}-a$.

* 6. Sum the series, each to n terms—

$$\text{(i.)} \cos a+\cos 2a+\cos 3a+ \ldots\ldots$$

$$\text{(ii.)} \tan a+2\tan 2a+2^2\tan 2^2a+ \ldots\ldots$$

* 7. Enunciate and prove De Moivre's Theorem.

Find all the roots of the equation $x^{2n}-2x^n \cos n\theta+1=0$.

* 8. If n be an even integer, investigate an expression for $\sin^n \theta$ in a series of cosines of multiples of θ.

LXVII.

1. Express in degrees the measure of an angle that is subtended by the arc of a circle equal in length to the radius. Show by what steps we obtain $\frac{\text{arc}}{\text{rad.}}$ as the measure of an angle. What unit is referred to in this measurement? For what measure of an angle and under what condition may we assume $\sin\theta=\theta$? Find $\sin 1'$; with what characteristic would its logarithm be registered in the tables?

2. Show that the numerical magnitude of the sine of an angle is limited and the tangent unlimited, whether positive or negative. Show that the tangent and cotangent of a given angle have always the same sign.

* Given the equations—

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\sqrt{1+\sin A}$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{1-\sin A}$$

express the values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ with proper signs to the quantities under the radicals, when A lies between 270° and 360°.

3. Prove the following formulæ :—

$$\text{(i.)}\ \frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}}.$$

$$\text{(ii.)}\ \cos 3A=4(\cos A)^3-3(\cos A).$$

From (ii.) deduce the value of $\sin 3A$ in terms of $\sin A$, and thence the value of $\tan 3A$ in terms of $\tan A$.

4. If A, B, C be the angles of a plane triangle, and a, b, c the sides subtending them respectively, prove—

$$*\ \text{(i.)}\ \sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

$$\text{(ii.)}\ \tan\frac{A}{2}=\sqrt{\frac{(a+c-b)(a+b-c)}{(a+b+c)(b+c-a)}}.$$

$$*\ \text{(iii.)}\ (a+b)\cos C+(b+c)\cos A+(c+a)\cos B=a+b+c.$$

5. Find the area of a regular polygon of n sides inscribed in a circle, and show, by increasing the number of sides of the polygon without limit, how the expression for the area of the circle may be obtained.

6. In a triangle ABC, given $AC=150$ feet, $BC=200$ feet, $AB=250$ feet; find all the angles of the triangle.

7. Two straight railroads are inclined to each other at an angle of $22^\circ\ 15'$. At the same instant two locomotives A and B start from a station at the point of their intersection, A travelling at the rate of 20 miles an hour along one road, and B travelling uniformly on the other; after three hours A and B are found to be 25 miles apart; show that there are two rates at which B may travel to fulfil this condition, and find those rates.

LXVIII.

1. Define the circular measure of an angle, and point out what is the unit in this measure. Find the circular measure of 80°, and the degrees in the angle of which the circular measure is 4.

* 2. Define the sine, cosine, and tangent of an angle of any magnitude; and prove that $\sin A = \sin(2n\ 180^\circ + A) = \sin\{(2n+1)\ 180^\circ - A\}$, n being integral. Find all the positive values of A less than two right angles which satisfy the equation $\sin 5A = \sin 11A$.

3. Prove the following—

$$(\alpha)\ \tan A = \sqrt{\frac{1-\cos 2A}{1+\cos 2A}}.$$

$$(\beta)\ \cos A \cos B = \cos^2\frac{A-B}{2} - \sin^2\frac{A+B}{2}.$$

$$(\gamma)\ \sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}.$$

4. Investigate the numerical values of sin 15° and sin 18°. Compare the area of the regular pentagon and the regular decagon inscribed in the same circle.

* 5. If a triangle is to be solved from given parts A, a, b, show that the solution is sometimes ambiguous; and that in such a case the difference of the two values of C is $2\cos^{-1}\dfrac{b\sin A}{a}$.

6. Express the radii of the four circles which touch the sides of a given triangle in terms of its sides. If the sides of the inscribed triangle are 51, 68, and 85 feet, show that the shortest side is divided by the point of contact of the inscribed circle into two segments one of which is double of the other.

7. In a plane triangle, having given two sides and the included angle, obtain the formulæ for solving the triangle.

Example— $a = 205$, $b = 195$, $C = 4°$,
$\log_{10} 2 = ·30103$ $\qquad$ $L\cot 2° = 11·4569162$
$L\cot 54°\,20' = 9·8549376$ $\qquad$ $L\cot 54°\,21' = 9·8546708.$

Find the remaining angles; and explain why in general, when $a = b$ nearly, the usual solution is not trustworthy, and why it may be employed in the case proposed.

* 8. What is meant by the base of a system of logarithms? If a, b, c be any three numbers, then $\log_a b \,.\, \log^c a = \log_c b$.

* 9. Assuming the expansion of $\log(1+x)$ in a series of ascending powers of x, show how to express the logarithm of a number in terms of the logarithm of the next inferior number, and of a rapidly converging series. Explain fully the use of this formula.

10. Investigate and explain the exponential values of the sine and cosine of an angle. Apply them to find the sum of n terms of the series,

$$\cos^4 a + \cos^4(a+\beta) + \cos^4(a+2\beta) + \text{etc.}$$

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(Questions included in the Higher Papers.)

LXIX.

* 1. Find all the values of θ which satisfy the equation—

$$\sin^2 n\theta - \sin (n-1)\theta = \sin^2 \theta.$$

2. Two similar and similarly situated triangles are such that the same circle can be described in the one and about the other; compare their areas, and prove that, if $A_1 = 4A_2$, the triangles must be equilateral.

3. Assuming De Moivre's Theorem, express $\sin a$ and $\cos a$ in ascending powers of a.

Sum the series $\cos a + \frac{1}{2}\cos 2a + \frac{1}{3}\cos 3a + \text{etc.}$

4. Find x from the equation—

$$3\tan^{-1}(x+1) = 2\tan^{-1}(x-1) + \tan^{-1}\frac{x}{2-x}\cdot$$

5. If $e^{\theta\sqrt{-1}}$ is a root of the equation $x^3 - px^2 + qx - r = 0$, prove that $\sin 3\theta - p \sin 2\theta + q \sin \theta = 0$.

LXX.

1. Prove that $\tan 3A = \tan\left(\frac{\pi}{3} - A\right) \tan A \, \tan\left(\frac{\pi}{3} + A\right)\cdot$

2. The circumference of a semicircle is divided into two arcs such that the chord of one is double that of the other. Show that the sum of the areas of the two segments cut off by these chords : the area of the semicircle :: 27 : 55. $\left(\pi = \frac{22}{7}\cdot\right)$

* 3. Prove that the expression $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{m}{n}}$, where m and n are positive integers, has n different values and no more.

Express in the form $\alpha+\beta\sqrt{-1}$ each of the roots of the equation $x^4+1=0$.

4. Find the sum of n terms of the series—

$$\sin\frac{\pi}{2n+1}-\sin\frac{3\pi}{2n+1}+\sin\frac{5\pi}{2n+1}-\text{etc.}\ldots$$

and expand $\log(\sec\theta)$ in a series of powers of $\tan\theta$.

LXX.*

1. Solve the equation—

$$1-\cos 2\theta=2(\cos a\cos\theta-\cos 2a).$$

2. Prove that—

$$\left(1-\tan^2\frac{\phi}{2}\right)\left(1-\tan^2\frac{\phi}{2^2}\right)\left(1-\tan^2\frac{\phi}{2^3}\right)\ldots\text{ to }\infty=\frac{\phi}{\tan\phi}.$$

3. Show that the perimeter of a triangle : perimeter of the inscribed circle :: area of the triangle : area of the circle

$$::\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}:\pi.$$

4. Investigate the expression for $\cos n\theta$ in a series of descending powers of $\cos\theta$, when n is a positive integer.

LXXI.

* 1. If θ be the circular measure of a positive angle less than a right angle, prove that $\sin\theta$ is $>\theta-\frac{\theta^3}{4}$; and that $\cos\theta$ is $<1-\frac{\theta^2}{2}+\frac{\theta^4}{16}$.

Given $\frac{\sin\theta}{\theta}=\frac{1013}{1014}$, prove that the value of θ is 4° 24′ nearly.

* 2. Expand $\sin^{10}\theta$ in terms of cosines of multiples of θ; and prove that $\log_e\left[\sqrt{2}\cos\frac{\theta}{2}\right]=\frac{1}{2}\cos\theta-\frac{1}{4}\cos^2\theta+\frac{1}{6}\cos^3\theta-\text{etc.}\ldots$

3. Expand θ in terms of $\tan\theta$; and prove that—

$$\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}.$$

LXXII.

1. Find the limit of the fraction $\frac{\sin\theta}{\theta}$, when θ is indefinitely diminished. Also if $\sin\phi=2\sin(\theta-\phi)$, find the limiting value of $\frac{\sin\theta}{\sin\phi}$ when θ and ϕ approach to π.

* 2. Assuming the truth of De Moivre's Theorem, express the cosine of an angle in a series of powers of its circular measure.

Show that the series is convergent.

3. The circumference of a semicircle of radius a is divided into n equal arcs. Show that the sum of the distances of the several points of section from either extremity of the diameter of the semicircle is equal to $a\left[\cot\frac{\pi}{4n}-1\right]$.

LXXIII.

* 1. Prove De Moivre's Theorem for positive integers. Find by means of it the values of $(-1+\sqrt{-3})^5+(-1-\sqrt{-3})^5$.

2. Prove that $\frac{\sin\theta}{\theta}=\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots$ to infinity.

* 3. Show that $\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\ldots$

LXXIV.

* 1. If $u=\cot^{-1}(\cos a)^{\frac{1}{2}}-\tan^{-1}(\cos a)^{\frac{1}{2}}$, prove that—

$$\sin u=\left(\tan\frac{a}{2}\right)^2.$$

* 2. Prove that $\cos a=1-\frac{a^2}{\underline{|2}}+\frac{a^4}{\underline{|4}}-$ etc.

If $\cos(a+\beta\sqrt{-1})=a+b\sqrt{-1}$
$\sin(a+\beta\sqrt{-1})=c+d\sqrt{-1}$, find a and β.

* 3. If a, b, c be the sides of a triangle, and C the circular measure of the angle opposite to c, prove that—

$$\log c=\log a-\frac{b}{a}\cos C-\frac{b^2}{2a^2}\cos 2C-\frac{b^3}{3a^3}\cos 3C-\text{etc.}$$

4. Sum to n terms—

(i.) $\sin\theta\sin 3\theta+\sin 2\theta\sin 6\theta+\sin 4\theta\sin 12\theta+\text{etc.}\ \ldots$

(ii.) $\cos^3 a+\cos^3(a+\beta)+\cos^3(a+2\beta)+\text{etc.}\ \ldots$

LXXV.

1. If A_1, B_1, C_1 be the angles of the triangle formed by the centres of the escribed circles of the triangle ABC, prove that $2A_1$, $2B_1$, $2C_1$ are the supplements respectively of A, B, and C. If the triangle $A_2B_2C_2$ be similarly formed from $A_1B_1C_1$, $A_3B_3C_3$ from $A_2B_2C_2$, and so on, find A_n, B_n, and C_n, and prove that, as n increases indefinitely, the triangle $A_nB_nC_n$ ultimately becomes equilateral.

2. Through the angular point C of a triangle ABC a straight line CPQ is drawn, on which are let fall the perpendiculars AP, BQ; prove that $PQ=AP\cot B\backsim BQ\cot A$.

3. Prove that $2\tan^{-1}\left[\tan\frac{a}{2}\tan\frac{x}{2}\right]=\cos^{-1}\left[\frac{\cos a+\cos x}{1+\cos a\cos x}\right]$.

* 4. Prove De Moivre's Theorem for a positive integral index, and for a negative integral index.

If $\sin(a+\beta\sqrt{-1})=x+y\sqrt{-1}$, prove that $x^2\operatorname{cosec}^2 a-y^2\sec^2 a=1$.

5. Find the sum of the series—

*(i.) $\sin a+\sin 2a+\sin 3a+\ \ldots$ to n terms.

(ii.) $\sin a+n\sin 2a+\frac{n(n-1)}{\underline{|2}}\sin 3a+\frac{n(n-1)(n-2)}{\underline{|3}}\sin 4a+\ldots$ to infinity.

LXXVI.

1. If the radius of the circumscribed circle of a triangle be equal to the diameter of the inscribed circle, show that the triangle is equilateral.

2. A circle is described touching the side BC of a triangle ABC, and the other two sides AB and AC, produced. Show that the same circle will be the inscribed circle of a triangle $AB'C'$ similar to ABC, formed by producing the sides AB and AC, and that the sides of the triangle $AB'C'$ will be—

$$\frac{sa}{s-a},\ \frac{sb}{s-b},\ \frac{sc}{s-c}\ \text{respectively, where } s=\frac{a+b+c}{2}.$$

3. In a triangle the least angle is 45°, and the tangents of the angles are in A. P. If the area be 3 square yards, find the lengths of the sides in feet, and express the other angles in the form $\tan^{-1}x$.

* 4. Assuming De Moivre's Theorem, prove that—

$$\cos a=1-\frac{a^2}{\underline{|2}}+\frac{a^4}{\underline{|4}}-\frac{a^6}{\underline{|6}}+\ldots$$

$$\sin a=a-\frac{a^3}{\underline{|3}}+\frac{a^5}{\underline{|5}}-\frac{a^7}{\underline{|7}}+\ldots$$

* 5. Find an expression for $\tan^{-1}x$ in a series of ascending powers of x; and hence show how to approximate to a value of π.

LXXVII.

1. Having given $\log_{10}2=\cdot301030$, and $\log_{10}7=\cdot845098$, find $\log_{10}14$, $\log_{10}98$, and $\log_{1000}\sqrt{\frac{8}{343}}$.

* 2. Prove that $\cos 30°+\cos 60°+\cos 120°+\cos 270°=\frac{1}{2}$; and that—

$$\frac{\sin 3\theta+2\sin 5\theta+\sin 7\theta}{\sin 5\theta+2\sin 7\theta+\sin 9\theta}=\frac{\sin 5\theta}{\sin 7\theta}.$$

3. If $A+B+C=90°$, prove that—

$$\tan B\tan C+\tan C\tan A+\tan A\tan B=1.$$

Hence show that—

$$2[\tan^2 A+\tan^2 B+\tan^2 C]-2=[\tan B-\tan C]^2+[\tan C-\tan A]^2 +[\tan A-\tan B]^2,$$

and that the expression $\tan^2 A+\tan^2 B+\tan^2 C$ is never less than unity.

4. Explain the method of computing the value of $\pi = 3{\cdot}1415926\ldots$ and calculate $e = 2{\cdot}7182818.\ldots$

5. Prove the following trigonometrical formulæ—

$$\text{(i.)}\ \cos na = \cos^n a - \frac{n(n-1)}{\underline{|2}} \cos^{n-2} a \sin^2 a + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}} \cos^{n-4} a \sin^4 a - \ldots$$

$$*\ \text{(ii.)}\ \sin a + \sin 2a + \ldots + \sin na = \frac{\sin \frac{n+1}{2} a \sin \frac{na}{2}}{\sin \frac{a}{2}}.$$

LXXVIII.

1. Find the general formula for all angles possessing the same sine as a given angle. Solve the equation $2 \sin a \sin 3a = \sin^2 2a$.

2. Establish the following—

$$*\ \text{(i.)}\ \frac{1+\cot 60^\circ}{1-\cot 60^\circ} = \left\{ \frac{1+\cos 30^\circ}{1-\cos 30^\circ} \right\}^{\frac{1}{2}}.$$

$$\text{(ii.)}\ \sin^{-1} \left\{ \frac{x-a+\beta}{2\beta} \right\}^{\frac{1}{2}} = \frac{1}{2} \cos^{-1} \frac{a-x}{\beta};$$

and that in any plane triangle $\log a = \log (b-c) + \mathrm{L} \cos \frac{A}{2} - \mathrm{L} \cos \phi$, where $\phi = \tan^{-1} \left\{ \frac{b+c}{b-c} \tan \frac{A}{2} \right\}$.

3. If O be the centre, and R the radius of the circle circumscribing a triangle; I the centre, and r the radius of an escribed circle; show that—

$$OI^2 = R^2 + 2Rr.$$

If D, E be the points in the base BC of the triangle ABC, where the inscribed and escribed circles respectively touch it, prove that—

$$BE = CD = s - c,\ BD = CE = s - b, \text{ where } s = \frac{1}{2}(a+b+c).$$

Hence prove that $rr_1 = (s-b)(s-c)$, where r, r_1 are the radii of the two circles.

* 4. Assuming De Moivre's Theorem for the case of a positive integral exponent, prove that it holds also for negative and fractional exponents.

Show that—

$$\tan(\alpha+\beta+\gamma+\ .\ .\ .)=\frac{s_1-s_3+s_5-\ .\ .\ .}{1-s_2+s_4-\ .\ .\ .}$$

when s_r denotes the sum of the products of $\tan\alpha$, $\tan\beta$, etc. r at a time.

* 5. Resolve $x^{2n}-2x^n\cos\theta+1$ into quadratic factors, and show that—

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ .\ .\ .$$

LXXIX.

1. Prove that—

(i.) $\sec^2\frac{\pi-\alpha}{4}+\sec^2\frac{\pi+\alpha}{4}=\sec^2\frac{\pi-\alpha}{4}\sec^2\frac{\pi+\alpha}{4}$.

(ii.) $\sec A+\sec(120°+A)+\sec(120°-A)+3\sec 3A=0$.

2. If O be the centre of the inscribed circle of a triangle ABC, prove that $a\ .\ AO^2+b\ .\ BO^2+c\ .\ CO^2=abc$.

* 3. Prove that $\cos\theta+\sqrt{-1}\sin\theta=e^{\theta\sqrt{-1}}$. Hence prove that—

$$e^\theta(\cos\theta+\sqrt{-1}\sin\theta)=e^{\theta\sqrt{2}\left(\cos\frac{\pi}{4}+\sqrt{-1}\sin\frac{\pi}{4}\right)}.$$

* 4. Find $\cos^{2m}\theta$ in cosines of descending multiples of θ.

Prove that $\cos^6\theta+\sin^6\theta=\frac{1}{8}(5+3\cos 4\theta)$.

5. Find the sum of n terms of each of the series—

* (i.) $\cos\theta+\cos 3\theta+\cos 5\theta+\ .\ .\ .$

(ii.) $\cos\theta\cos 2\theta+\cos 2\theta\cos 3\theta+\cos 3\theta\cos 4\theta+\ .\ .\ .$

6. Find $\cos 3\alpha$ in terms of $\cos\alpha$. Prove that $2\sqrt{2}\cos\alpha$ is one of the roots of the equation $x^3-6x-4\sqrt{2}\cos 3\alpha=0$.

LXXX.

1. Prove that the circumferences of circles vary as their radii. For what purpose does Trigonometry avail itself of this property of circles?

* 2. If a, b, c, the sides of a triangle, be in arithmetical progression, prove that $2\sin\frac{A}{2}\sin\frac{C}{2}=\sin\frac{B}{2}$, A, B, C being the angles subtended by a, b, c respectively.

3. Given $\cos(\alpha-\beta)\cos\alpha = a\sin\beta\cos\beta$, $\sin(\alpha-\beta)\cos\alpha = \frac{a}{2}\cos^2\beta$, prove that $a = \frac{2\sin\beta}{1+3\sin^2\beta}$.

4. Solve the equation—

$$5^x + 5^{-x} = 125{\cdot}008\text{; given, if required, } \log 2 = {\cdot}301030.$$

* 5. Show that—

$\cos\frac{p\theta}{q} + \sqrt{-1}\sin\frac{p\theta}{q}$ is one of the values of $(\cos\theta+\sqrt{-1}\sin\theta)^n$ where $n = \frac{p}{q}$. Hence find the values of $(-1)^{\frac{1}{3}}$.

* 6. Find the sum of the series—

$$\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots \text{ to } n \text{ terms.}$$

LXXXI.

1. If $\tan\theta + \tan 2\theta + \tan 3\theta = 0$, find θ.

* 2. Show that $\sin 9°$ lies between ·156 and ·157.

3. Eliminate α and β from the equations—

$$x = (a\sin^2\alpha + b\cos^2\alpha)\cos^2\beta + e\sin^2\beta.$$
$$y = a\cos^2\alpha + b\sin^2\alpha.$$
$$z = (b-a)\sin\alpha\cos\alpha\cos\beta.$$

4. The centres of the escribed circles of a given triangle are joined; find the area of the triangle thus formed in terms of the sides of the given triangle.

5. Prove that the area of a circle is given by the expression πr^2, where r is the radius of the circle and π is twice the circular measure of a right angle.

* 6. Assuming that $(\cos\theta+\sqrt{-1}\sin\theta)^p = \cos p\theta + \sqrt{-1}\sin p\theta$, when p is a positive integer, prove that the same formula holds when p is negative, and also when p is fractional.

In the case where $p = \frac{1}{n}$, show how to get the rest of the nth roots of $\cos\theta + \sqrt{-1}\sin\theta$.

LXXXII.

* 1. What are the logarithms to base 3 of 81, $\frac{1}{81}$, and $\sqrt[4]{729}$? How many positive integers are there whose logarithms to base 3 have 2 for a characteristic?

2. Three equal circles touch each other. Find the radius of the circle circumscribing the three circles.

3. AD, BE, CF are the perpendiculars from the angular points of a triangle ABC upon the opposite sides; prove that the sides of the triangle DEF are respectively equal to $a \cos A$, $b \cos B$, $c \cos C$.

4. A man stands on the top of a wall of height h, and observes the angular elevation a of the top of a telegraph post; he then descends from the wall, and finds that the angular elevation is now β; prove that the height of the post exceeds the height of the man by $h\frac{\sin\beta\cos a}{\sin(\beta-a)}$.

* 5. Solve the equation $\sin 5x \cos 3x = \sin 9x \cos 7x$; and reduce $\cos(\theta+\phi\sqrt{-1})$ to the form $a+\beta\sqrt{-1}$.

* 6. Prove that $2^6(\cos^8 a+\sin^8 a)=\cos 8a+28\cos 4a+35$, and find the values of θ between 0 and $\frac{\pi}{2}$ which satisfy the equation—

$$\sin^2\theta+\cos^2 2\theta=\frac{3}{4}.$$

7. Two towers, whose heights respectively are 180 and 80 feet, stand on a horizontal plane; from the foot of each column the angle of elevation of the other is taken, and one angle is found to be double the other; prove that the horizontal distance between the towers is 240 feet, and show that the sine of the greater angle of elevation is $\frac{3}{5}$.

8. If r be the radius of a circle and πr^2 its area, what ratio does the constant π express? Prove $\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$; point out, without performing the calculation, how the numerical value of π may be thus determined.

* 9. If $2\cos\theta=x+\frac{1}{x}$, prove $2\cos m\theta=x^m+\frac{1}{x^m}$.

Prove also, if m be a whole number,

$$2^{m-1}(\cos\theta)^m=\cos m\theta+m\cos(m-2)\theta+\frac{m(m-1)}{1.2}\cos(m-4)\theta+\text{etc.}$$

Write down the last terms of this series when m is odd.

LXXXIII.

1. If $\sin a = \tan \beta$, prove that—

$$\sin (a - \beta) \cos \beta = \sin 2\beta \sin^2 \frac{a}{2}.$$

2. Having given $\log_{10} 2 = \cdot 301030$ and $\log_{10} 7 = \cdot 845098$, find $\log_{10} 98$ and $\log_{1000} \frac{343}{8}$.

3. If the distance between two stations C and D on a horizontal plane subtend equal angles (a) at two other stations A and B on the plane, and if s be the sum of the angles ABD, BAC, prove that—

$$CD = AB \frac{\sin a}{\sin (s - a)}.$$

4. Find the relation between the sides of a triangle ABC, when—

$$\text{(i.)} \quad \sin B \sin C = \cos^2 \frac{A}{2}.$$

$$\text{(ii.)} \quad \sin B \sin C = \frac{\sin^2 A}{2}.$$

5. If s be the sum of the sides of a triangle, a and A any side and the opposite angle, the area of the triangle is $s(s-a) \tan \frac{A}{2}$.

If a regular octagon be described about a circle, and a similar octagon be inscribed in the circle, the area of the described figure is to that of the inscribed figure as $2\sqrt{2}\,(\sqrt{2} - 1) : 1$.

6. Show that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \text{etc.}$

The quantity $\sin^{-1} x + \cos^{-1} x$ is independent of x.

7. Sum to $2n$ terms the series—

$$\cos a + \cos (a + 2\beta) + \cos (a + 4\beta) + \ldots$$

8. Two poles, a and b feet long respectively, are placed vertically on a horizontal plane so as to subtend the angle a at a point in the line joining their feet. If β, β', be the angles which they subtend at any point in the horizontal plane at which the line joining their feet subtends a right angle, then

$$(a + b)^2 \cot^2 a = a^2 \cot^2 \beta + b^2 \cot^2 \beta'.$$

LXXXIV.

1. Given $\log_8 9 = a$ and $\log_3 5 = b$,
Find log 2 and log 3 to base 10 in terms of a and b.

2. If $\tan\theta = \frac{\sin a \cos\beta}{\sin\beta + \cos a}$, prove that $\tan\frac{\theta}{2} = \frac{(1-\sin\beta)(1-\cos a)}{\sin a \cos\beta}$

$$\text{and that } \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{\cos\left(\frac{\pi}{4} - \frac{a+\beta}{2}\right)}{\sin\left(\frac{\pi}{4} - \frac{a-\beta}{2}\right)}.$$

3. If $2s = a + b + c$, prove that $(b-c)(s-a)^2 = s^2(b-c) - a(b^2 - c^2)$; and prove that in any triangle ABC whose sides are a, b, c—

$$(b-c)(s-a)\cos^2\frac{A}{2} + (c-a)(s-b)\cos^2\frac{B}{2} + (a-b)(s-c)\cos^2\frac{C}{2} = 0.$$

4. If Δ be the area of a quadrilateral inscribed in a circle, show that the diagonals meet at the angle $\sin^{-1}\frac{2\Delta}{ac+bd}$ where a, b, c, d are the sides of the quadrilateral.

If circles can be both inscribed in and described about the quadrilateral, show that the angle between the diagonals is $\cos^{-1}\frac{ac \backsim bd}{ac+bd}$.

5. Prove from De Moivre's Theorem, or otherwise, that—

$$\log\frac{\cos n\theta}{\cos^n\theta} = \frac{n\tan^2\theta - \tan^2 n\theta}{2} - \frac{n\tan^4\theta - \tan^4 n\theta}{4} + \text{etc.}$$

* 6. Show that $\tan(a + \beta + \gamma + \ldots) = \frac{s_1 - s_3 + s_5 - \text{etc.}}{1 - s_2 + s_4 - \text{etc.}}$ where s_r denotes the sum of the products of the tangents of the angles a, β, γ . . . taken r together.

7. Sum the series—

* (i.) $\sin a - \sin(a+\beta) + \sin(a+2\beta) - \text{etc.} \ldots$

(ii.) $\sin a + \sin\beta \sin 2a + \sin^2\beta \sin 3a + \ldots$

each to n terms.

* 8. Show how to resolve the expression $x^{2n} - 2x^n \cos\theta + 1$ into quadratic factors.

LXXXV.

1. If $3\sin^2 A + 2\sin^2 B = 1$, and $3\sin 2A - 2\sin 2B = 0$, prove that—

$$\sin(B-A) = \frac{1}{9}\sqrt{6}.$$

2. If R be the radius of the circumscribing circle, s the semi-perimeter, and S the area of the triangle whose sides are a, b, c, prove that—

$$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4R}{S}.$$

3. If a, b, c, d, be the sides of a quadrilateral inscribed in a circle, and if $2s = a + b + c + d$, prove that the area of the quadrilateral is equal to $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

4. Assuming De Moivre's Theorem for integral indices, prove it for fractional indices.

By aid of this theorem find all the fifth roots of unity.

* 5. If n be a positive integer, expand $(\cos\theta)^{2n+1}$ in a series of cosines of multiples of θ.

* 6. Write down the exponential expression for $\cos\theta$.

Prove $2\cos(n\cos^{-1}x) = (x + \sqrt{-1}\sqrt{1-x^2})^n + (x - \sqrt{-1}\sqrt{1-x^2})^n$.

7. Find the sum of—

(i.) $\sin\theta - \dfrac{\sin 2\theta}{\underline{|2}} + \dfrac{\sin 3\theta}{\underline{|3}} - \ldots$ to infinity.

* (ii.) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots$

(iii.) $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots$

LXXXVI.

1. If α and β be two values of θ satisfying the equation—

$$\frac{\cos\theta}{a}+\frac{\sin\theta}{b}=\frac{1}{c}, \text{ prove that}$$

$$a\cos\frac{\alpha+\beta}{2}=b\sin\frac{\alpha+\beta}{2}=c\cos\frac{\alpha-\beta}{2}.$$

2. Equilateral triangles DBC and EBC are described upon opposite sides of BC, one of the sides of the triangle ABC. Prove that

$$AD^2+AE^2=AB^2+AC^2+BC^2.$$

* 3. If in the triangle ABC the angle A be three times the angle B, prove that $\sin B=\frac{1}{2}\sqrt{\frac{3b-a}{b}}$.

4. In any plane triangle of which A, B, C are the angles, and a, b, c the sides subtending them, prove—

$$\text{(i.) } c^2=(a+b)^2\sin^2\frac{C}{2}+(a-b)^2\cos^2\frac{C}{2}.$$

$$\text{(ii.) } \sqrt{bc\sin B\sin C}=\frac{b^2\sin C+c^2\sin B}{b+c}.$$

* 5. Prove that $\log(x+y\sqrt{-1})=\frac{1}{2}\log(x^2+y^2)+\tan^{-1}\frac{y}{x}\sqrt{-1}$.

6. If $x+\frac{1}{x}=2\cos\theta$, and $y+\frac{1}{y}=2\cos\phi$, prove—

$$\text{(i.) } x^m+\frac{1}{x^m}=2\cos m\theta.$$

$$\text{(ii.) } x^my^n+\frac{1}{x^my^n}=2\cos(m\theta+n\phi).$$

7. Investigate the expression—

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\text{ etc.,}$$

and hence prove—

$$\pi=2\,\frac{2^2}{1.3}\cdot\frac{4^2}{3.5}\cdot\frac{6^2}{5.7}\cdot\frac{8^2}{7.9}\text{ etc.}$$

* 8. Find the sum of *either* of the two following series—

(i.) $\sin\theta+\sin 2\theta+\sin 3\theta+\ .\ .\ .\ +\sin n\theta.$

(ii.) $\cos\theta+\cos 2\theta+\cos 3\theta+\ .\ .\ .\ +\cos n\theta.$

LXXXVII.

1. Define a logarithm, and prove that $\log_a b \times \log_b a = 1$. Having given $\log_{10} 2 = \cdot 30103$ and $\log_{10} 49 = 1\cdot 690196$, find $\log_{10} \cdot 0056$ and $\log_{100} 1120$.

2. Through the centre O, of the circumscribing circle of a triangle ABC, AOD is drawn to meet BC in D; prove that

$$OD : BD : CD :: \cos A : \sin 2C : \sin 2B.$$

3. Enunciate and prove De Moivre's Theorem. Apply this theorem to find all the fifth roots of unity.

* 4. Obtain the exponential values of $\sin \theta$ and $\cos \theta$.

If $\tan \theta = n \tan \alpha$, prove that

$$\theta = \alpha - \frac{1-n}{1+n} \sin 2\alpha + \frac{1}{2}\left(\frac{1-n}{1+n}\right)^2 \sin 4\alpha - \ldots\ldots$$

* 5. Prove that the infinite series

$$\cos \theta - \frac{1}{2} \cos 2\theta + \frac{1}{3} \cos 3\theta - \frac{1}{4} \cos 4\theta + \ldots = \log\left(2 \cos \frac{\theta}{2}\right).$$

LXXXVIII.

1. Define a logarithm, and prove that $\log_a N = \dfrac{\log_b N}{\log_b a}$.

Prove that if $\log_a b = m$ and $\log_b a = n$, then $\left\{\dfrac{\log_a m}{\log_b n}\right\}^2 = \dfrac{m}{n}$.

2. If $\tan^2 \theta - \sec^2 \alpha = 1$, prove that $\sec \theta + \tan^3 \theta \operatorname{cosec} \theta = (3 + \tan^2 \alpha)^{\frac{3}{2}}$.

* 3. If the sides of a triangle be in A. P. the cotangents of its semi-angles will also be in A. P.

4. A man walking towards a tower, on the edge of which a flagstaff is fixed, observes that when his distance from the tower is c feet, the flagstaff subtends the greatest angle at his eye, and he finds that this angle is α; prove that the height of the flagstaff is $2c \tan \alpha$.

5. Find the expansion of $\log_e (1 + x)$ in ascending powers of x.

Prove that the coefficient of x^n in the expansion of $\log(1 + x + x^2)$ is $-\dfrac{2}{n}$ or $+\dfrac{1}{n}$, according as n is or is not a multiple of 3.

6. Prove that $\cos a + \sqrt{-1}\,\sin a = e^{a\sqrt{-1}}$; hence prove that

$$\cos\left(\frac{\pi}{4} + \theta\sqrt{-1}\right) = \frac{1}{2\sqrt{2}}[(e^{\theta} + e^{-\theta}) - \sqrt{-1}(e^{\theta} - e^{-\theta})]$$

* 7. If θ be the circular measure of an angle, prove that

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ .\ .\ .\ .\ \text{and that}$$

$$(3^{\frac{1}{2}} - 3^{-\frac{1}{2}}) - \frac{1}{3}(3^{\frac{3}{2}} - 3^{-\frac{3}{2}}) + \frac{1}{5}(3^{\frac{5}{2}} - 3^{-\frac{5}{2}}) - \ .\ .\ .\ .\ = \frac{\pi}{6}.$$

* 8. Find an expression for $\cos^n\theta$ in terms of cosines of descending multiples or θ, when n is an even integer.

LXXXIX.

1. If a^x can be expanded in an infinite series of the form $1 + Ax + Bx^2\ .\ .\ .\ .$; find A. Deduce the expansion of $\log_e(1+x)$.

* Prove that $\log\frac{3}{2} = \frac{1}{2} - \left[\frac{2}{1.3.2^3} + \frac{3}{2.5.2^5} + \ .\ .\ .\ \right]$

2. Prove De Moivre's Theorem for all forms of the index.

* 3. Find the exponential expressions for $\sin\theta$ and $\cos\theta$, and prove that $(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{\pi}{2}}$.

XC.

1. Prove that $\cos x \cos(a - x)$ has its greatest value when $x = \frac{a}{2}$. Hence prove that the product of the cosines of the three angles of a triangle cannot be greater than $\frac{1}{8}$.

2. Prove that if $\sin\alpha$ and $\sin\beta$ be the two values of $\sin\theta$ satisfying the equation $a\cos 2\theta + b\sin 2\theta = c$, then $\cos^2\alpha - \sin^2\beta = \frac{ac}{a^2 + b^2}$.

3. If R be the radius of the circle circumscribing the triangle ABC, whose sides are a, b, and c, prove that $\tan A = \frac{1}{R} \cdot \frac{abc}{b^2 + c^2 - a^2}$.

Hence show that if the squares of the sides be in arithmetical progression, the tangents of the angles will be in harmonical progression.

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XCI.

1. Prove—

* (i.) $\tan \frac{15^\circ}{2} = (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1)$.

(ii.) $\frac{\cos 3\theta + 2 \cos 5\theta + \cos 7\theta}{\cos \theta + 2 \cos 3\theta + \cos 5\theta} = \cos 2\theta - \sin 2\theta \,.\, \tan 3\theta$.

2. If A, B, C be the angles of a plane triangle, find $\tan \frac{A}{2}$ in terms of the sides of the triangle.

Hence prove that $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{A}{2} \tan \frac{C}{2} + \tan \frac{B}{2} \tan \frac{C}{2} = 1$.

3. If a, b, c, d, be the sides of a quadrilateral, and θ, ϕ, ψ be the angles respectively at which a and b, a and c, b and c are inclined to each other, prove $d^2 = a^2 + b^2 + c^2 - 2ab \cos \theta - 2ac \cos \phi - 2bc \cos \psi$.

* 4. Two straight roads intersect at an angle of 30°; from the point of junction two pedestrians, A and B, start at the same time, A walking along one of the roads at the rate of 5 miles an hour, B walking uniformly along the other road. At the end of 3 hours A and B are 9 miles apart. Show that there are two rates at which B may walk to fulfil the condition, and determine the slower rate of the two. Given

$\log 3 = \cdot 4771213$ $\qquad$ $L \sin 56^\circ\, 26' = 9\cdot 9207717$

$\log 2 = \cdot 3010300$ $\qquad$ $L \sin 56^\circ\, 27' = 9\cdot 9208555$

$L \sin 26^\circ\, 26'\, 33'' = 9\cdot 6486522$ $\qquad$ $\log 8\cdot 0154 = \cdot 9039248$.

XCII.

1. Define the circular measure of an angle, and find the circular measure of a right angle.

Prove that the same angles will be indicated by each of the following formulæ:—

$$(2n - 1)\frac{\pi}{2} + (-1)^n \frac{\pi}{3} \text{ and } 2n\pi \pm \frac{\pi}{6},$$

n being integral.

2. Prove that—

$$\text{(i.)}\ \sin^2\left(\frac{\pi}{8}+\frac{\theta}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{\theta}{2}\right)=\frac{1}{\surd 2}\sin\theta.$$

$$*\ \text{(ii.)}\ 2\tan^{-1}\left(\tan\frac{a}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right)=\tan^{-1}\left(\frac{\sin a\cos\beta}{\sin\beta+\cos a}\right).$$

And if A and B be two acute angles, such that

$$3\sin^2 A+2\sin^2 B=1$$
$$3\sin 2A-2\sin 2B=0$$

prove that $A+2B=90°$.

3. If R be the radius of the circle circumscribing the triangle ABC, prove that—

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R.$$

If AD, BE, and CF be perpendiculars from A, B, and C upon the opposite sides respectively, prove that $DE=AB\cos C$, and also that the perimeter of the triangle DEF is $4R\sin A\sin B\sin C$.

4. Find exponential expressions for $\sin\theta$ and $\cos\theta$.

If A, B, and C be the angles of any triangle, and i be the unit of circular measure, and if—

$$x+\frac{1}{x}=2\cos A,\qquad y+\frac{1}{y}=2\cos B,\qquad z+\frac{1}{z}=2\cos C,$$
$$u+\frac{1}{u}=2\cos i,$$

prove that $\log_u(xyz)$ lies between $+\pi$ and $-\pi$.

5. Find the sum of—

$$\text{(i.)}\ \cos\frac{A}{2}+\cos\frac{3A}{2}+\cos\frac{5A}{2}+\text{etc., to } n \text{ terms.}$$

$$\text{(ii.)}\ \sin A+\frac{1}{2}\sin 2A+\frac{1}{2^2}\sin 3A+\text{ etc., to infinity.}$$

Apply the result of (i.) to prove that, if one angle of a triangle be n times another angle, the side opposite the former angle is less than n times the side opposite the latter angle, where n is any integer.

XCIII.

* 1. Having given $\sin A$, show how to find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$; and obtain a general expression for the values between which $\frac{A}{2}$ must be in order that $\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)$ and $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)$ may both be positive.

2. In any plane triangle whose sides are a, b, c, and angles A, B, C, prove that $a:b:c::\sin A:\sin B:\sin C$.

Also if $b-a=3c$, show that $A+\frac{C}{2}=\cos^{-1}\left(3\cos\frac{C}{2}\right)$ and that $3\sin B\cot\frac{B-A}{2}=1+3\cos B$.

3. Find the area of a quadrilateral, which can be inscribed in a circle, in terms of its sides.

If the sides taken in order be 3, 3, 4, 4 feet in length respectively, find the radii of the inscribed and circumscribed circles of the quadrilateral.

4. If the tangents of the angles of a plane triangle are in arithmetical progression, the squares on its sides will also form an arithmetical progression.

5. A tower with a flagstaff on it stands on a horizontal plane; show that the distances from its base at which the flagstaff subtends the same angle at the observer's eye, and that at which it subtends the greatest possible angle, form a geometrical progression.

6. Assuming De Moivre's Theorem for a positive integral index, prove it for a negative or fractional index.

In the last case show how many different values may be obtained; and find four values for the expression $(-1)^{\frac{1}{4}}$.

* 7. Sum the series $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots$ to n terms.

Deduce the sum of n terms of the series—

$$\sin^2\alpha+\sin^2 2\alpha+\sin^2 3\alpha+\ldots\ldots$$

XCIV.

1. Prove the formula—

$$\cos(A-B)=\cos A\cos B+\sin A\sin B,$$

and show that, if the tangent of one angle of a triangle be equal to the sum of the sines of the other two, the tangents of the halves of the angles will be in geometrical progression.

2. Find an expression for the area of a plane triangle in terms of its sides, and show that in an isosceles triangle whose least angle = 30°, the area is equal to one-fourth of the square described on one of the equal sides.

* 3. Explain the ambiguous case in the solution of plane triangles, and show that the circles which circumscribe the two triangles are equal.

4. Obtain all possible values of the expression—

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}.$$

Find (1) the product, (2) the sum of the n expressions obtained by giving r the several values 1, 3, 5, . . . $(2n-1)$ in—

$$\cos\frac{r\pi}{2n}+\sqrt{-1}\sin\frac{r\pi}{2n}\cdot$$

5. Resolve $x^{2n}-1$ into factors, n being any positive integer.

6. Prove that—

$$\tan\left(3\theta-\frac{\pi}{4}\right)\tan\left(\theta+\frac{\pi}{12}\right)=\tan\left(\theta+\frac{\pi}{4}\right)\tan\left(\theta-\frac{\pi}{12}\right).$$

7. If the centres of the inscribed and circumscribed circles of a triangle be at equal distances from one side, the cosines of the angles adjacent to that side will together be equal to 1.

8. The circumference of a circle is divided into twelve arcs, whose lengths, taken in order, are in arithmetical progression, and the first six together form a quadrant. Show that the area of the polygon bounded by the chords of the twelve arcs is $\frac{a^2}{8}$ cosec $\frac{\pi}{72}$, where a is the radius of the circle.

XCV.

1. Prove that in the common system of logarithms the characteristic of the logarithm of any number can be found by inspection.

* Find $\sqrt[5]{\cdot 002}$, having given $\log 5 = \cdot 69897$ and $\log 2\cdot 8854 = \cdot 460206$.

* 2. Give a geometrical proof of the formula—

$$\cos A + \cos B = 2 \cos \frac{1}{2} (A+B) \cos \frac{1}{2} (A-B).$$

Show that, so long as the sum of the cosines of the angles of a triangle is constant, the product of the sines of the halves of the angles will also be constant.

3. Prove that the sides of a triangle are proportional to the sines of the opposite angles.

A line drawn parallel to the hypothenuse of a triangle right-angled at C meets the sides CB, CA in P, Q respectively. Show that—

$$\cot PAB - \cot QBA = 2 \cot 2A.$$

4. Find the radii of the inscribed and circumscribed circles of a regular polygon.

The alternate angular points of a regular pentagon are joined by straight lines. Find the length of a side of the pentagon formed by these lines, and show that the radius of the circle circumscribing it is

$$a\sqrt{1 - \frac{2}{\sqrt{5}}},$$

where a is the side of the original pentagon.

* 5. Expand $(\cos\theta)^{2n+1}$ in a series of cosines of multiples of θ, and deduce the expansion of $(\sin\theta)^{2n+1}$ in a series of sines of multiples of θ.

Write down in each series the general term and the last term.

6. Find trigonometrical expressions for all the roots of the equation $x^n - 1 = 0$, n being a positive integer.

Show that the roots can always be arranged in a series of the form—

$$a, a^2, a^3 \;.\;.\;.\; a^n.$$

7. If $2\cos\theta - \cos 2\theta = x$, and $2\sin\theta - \sin 2\theta = y$, prove that—

$$(x^2 + y^2 - 3)^2 = 12 - 8x.$$

8. If A, B, C be the angles of a triangle, prove that—

$$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C.$$

If twice the square on the diameter of the circumscribing circle of the triangle is equal to the sum of the squares on the sides, prove that the triangle is right-angled.

9. Prove that $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$; and that

$$\cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \cos\frac{6\pi}{11} + \cos\frac{8\pi}{11} + \cos\frac{10\pi}{11} = -\frac{1}{2}.$$

10. If θ be less than $\frac{\pi}{4}$, prove that—

$$\tan^2\theta - \frac{\tan^4\theta}{2} + \frac{\tan^6\theta}{3} - \frac{\tan^8\theta}{4} + \ldots = \sin^2\theta + \frac{\sin^4\theta}{2} + \frac{\sin^6\theta}{3} + \ldots$$

Hence, or otherwise, prove that—

$$\frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{6} + \frac{1}{7}\cdot\frac{1}{8} + \ldots = \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{8} + \frac{1}{4}\cdot\frac{1}{16} + \frac{1}{5}\cdot\frac{1}{32} + \ldots$$

XCVI.

1. If the angle A be greater than the angle B, and less than a right angle, prove, by means of geometrical figures, the formulæ—

$$\cos(A - B) = \cos A \cos B + \sin A \sin B,$$
$$\cos B - \cos A = 2\sin\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B).$$

* Prove that $\tan 5A - \tan 3A - \tan 2A = \tan 5A \tan 3A \tan 2A$.

* 2. Find the sine and cosine of an angle of 18°, and prove that—

$$\sin 9^\circ = \frac{1}{4}\sqrt{3 + \sqrt{5}} - \frac{1}{4}\sqrt{5 - \sqrt{5}}.$$

3. If R be the radius of the circumscribed circle of a triangle ABC, prove that the distances between the centres of the escribed circles are respectively

$$4R\cos\frac{A}{2},\ 4R\cos\frac{B}{2},\ 4R\cos\frac{C}{2}.$$

If D, E, F be the centres of these circles, prove that the distances between the centres of the escribed circles of the triangle DEF are respectively

$$8R\cos\frac{B+C}{4},\ 8R\cos\frac{C+A}{4},\ 8R\cos\frac{A+B}{4}.$$

4. Enunciate and prove De Moivre's Theorem.

Find the five values of the expression $\left\{\frac{1+\sqrt{-3}}{2}\right\}^{\frac{1}{5}}$.

* If $\log(x+y\sqrt{-1})=a+\beta\sqrt{-1}$, prove that $x^2+y^2=e^{2a}$, and $y=x\tan\beta$.

* 5. Prove that the sum of n terms of the series,

$$\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta+\ \ldots \text{ is } \frac{\sin^2 n\theta}{\sin\theta},$$

and that the sum of the infinite series

$$\sin\theta+\frac{\sin 2\theta}{\underline{|2}}+\frac{\sin 3\theta}{\underline{|3}}+\ \ldots \text{ is } e^{\cos\theta}\sin(\sin\theta).$$

6. If $\frac{\sin\theta}{\cos\theta-\cos a}+\frac{\cos\theta}{\sin a-\sin\theta}=\frac{1}{\sin(a-\theta)}$, find the general value of θ.

7. Prove that the expression $a\cos^2\theta+2b\cos\theta\sin\theta+c\sin^2\theta$ (a) may be expressed in the form $\frac{a+c}{2}+\sqrt{m^2+n^2}\sin(2\theta+\phi)$, where $m=\frac{a-c}{2}$ $n=b$, $\tan\phi=\frac{m}{n}$. Hence prove that the greatest and least values of (a) are the roots of the equation $(x-a)(x-c)=b^2$.

* 8. Prove that $e^{\theta}(\cos\theta+\sqrt{-1}\sin\theta)=e^{\theta\sqrt{2}\left(\cos\frac{\pi}{4}+\sqrt{-1}\sin\frac{\pi}{4}\right)}$.

Hence prove that the coefficients of θ^{2n} in the expressions of $e^{\theta}\cos\theta$ and $e^{\theta}\sin\theta$ will be $\frac{2^n}{\underline{|2n}}\cos\frac{n\pi}{2}$ and $\frac{2^n}{\underline{|2n}}\sin\frac{n\pi}{2}$, respectively.

XCVII.

1. If a be the least positive angle which has a given sine, prove that the general expression for the angle is $n\pi+(-1^n)\,a$, where n is zero or any positive or negative integer.

Solve the equation $\sin\theta\cos\theta+\sin a\cos a=\sin(a+\theta)$.

* 2. Find $\tan\frac{A}{2}$ in terms of $\tan A$, and explain the existence of two values. Prove that $\sec^2 15°=4\tan 15°$, and that $\tan 7° 30'=\sqrt{6}+\sqrt{2}-2-\sqrt{3}$.

3. Prove that in any triangle ABC,

$$a \sin \frac{1}{2}(B-C) = (b-c)\cos\frac{A}{2}, \text{ and } a\cos\frac{1}{2}(B-C) = (b+c)\sin\frac{A}{2}.$$

Having given $A = 60°$, and $\frac{b}{c} = \frac{\sqrt{3}+1}{2}$, find the angles B and C.

4. If $(\sin^2\alpha - \sin^2\beta)(\sin^2\alpha - \sin^2\gamma) = \sin^2\beta\sin^2\gamma\cos^4\alpha$, prove that $\tan^2\alpha = \tan^2\beta + \tan^2\gamma$.

5. If $\frac{\log x}{x(y+z-x)} = \frac{\log y}{y(z+x-y)} = \frac{\log z}{z(x+y-z)}$, prove $x^y . y^x = y^z . z^y = z^x . x^z$.

6. Having given two sides and the included angle of a triangle, find the other angles and the third side.

$A = 54°$, $b = 2·25$ feet, $c = 1·75$ feet, find B and C, having given

$\log 2 = ·301030$	$L \tan 13° \ 47' = 9·389724$
$L \cot 27° = 10·292834$	$L \tan 13° \ 48' = 9·390270$.

If AD, BE, CF be the perpendiculars from the angular points of a triangle ABC upon the opposite sides, prove that the areas of the triangles DEF, ABC are in the ratio of $2\cos A\cos B\cos C : 1$.

7. If S be the area and s the semi-perimeter of a triangle whose sides are a, b, c, prove that $x^3 - x^2\left(\frac{abc}{S} + \frac{S}{s}\right) + xs^2 - Ss = 0$, where x is equal to the radius of any one of the escribed circles.

* 8. Write down the series for $\sin x$ and $\cos x$ in powers of x, and prove that $2\cos x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$, and $2\sqrt{-1}\sin x = e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}$.

Having given $(1+c)\tan x = (1-c)\tan y$, prove that

$$x = y - c\sin 2y + \frac{c^2}{2}\sin 4y - \frac{c^3}{3}\sin 6y + \text{etc.}$$

XCVIII.

1. If the logarithms of numbers be found to any one base, show how to find them to any other.

Given $\log_{10} 2 = ·30103$, find $\log(·000125)$ to the base $\sqrt{·05}$.

2. Express the sine and cosine of an angle in terms of its cotangent, and if $\operatorname{cosec}\theta + \cot\theta = a$, prove that $\sec\theta + \tan\theta = \frac{a+1}{a-1}$.

3. Find the following functions—

$$\sin 15°, \qquad \cos 22\tfrac{1}{2}°, \qquad \tan 7\tfrac{1}{2}°.$$

4. Solve the following triangles—

* (i.) $a=\sqrt{6}, \qquad b=2, \qquad c=\sqrt{3}+1.$

(ii.) $b=\sqrt{2}, \qquad c=\sqrt{3}+1, \qquad A=45°.$

5. Prove that if θ be the circular measure of an angle, the ratios $\frac{\sin\theta}{\theta}$ and $\frac{\cos\theta}{1}$ become ultimately equal to unity when the angle is infinitely diminished.

Given in a triangle, $b=100$, $c=99$, $B=60°$, find a correct to one decimal place, and A and C correct to minutes.

6. The sides of a quadrilateral figure, inscribed in a circle, taken in order, are 60, 25, 52, 39. Prove that the lengths of the diagonals are 65 and 63, and that the area is 1764.

* 7. State De Moivre's Theorem, and prove its truth in the case of a fractional index.

Employ it in finding all the roots of $x^6-1=0$.

* 8. Find the sum of the following series to n terms—

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ .\ .\ .$$
$$\sin\alpha+c\sin(\alpha+\beta)+c^2\sin(\alpha+2\beta)+\ .\ .\ .$$

XCIX.

* 1. Find expressions for $\tan\theta$ and $\sec\theta$ in terms of $\sin\theta$.

If $\sin\theta=\frac{a(a+2b)}{a^2+2ab+2b^2}$, find $\sec\theta$ and $\tan\theta$, and prove that—

$$\frac{a+b}{b}=\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right) \text{ or } -\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right).$$

* 2. Taking $A+B$ less than a right angle, prove the formulæ—

$$\sin(A+B)=\sin A\cos B+\cos A\sin B,$$
$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

* Prove also the relation

$$\frac{\sin A+\sin(n-1)A+\sin(2n-3)A}{\cos A+\cos(n-1)A+\cos(2n-3)A}=\tan(n-1)A.$$

3. If the two sides b, c, and the included angle A of a triangle ABC be given, show how to find the other angles. Prove that the side a is given by the equation $a=(b \backsim c)\sec\theta$, where $\tan^2\theta=\dfrac{4bc}{(b-c)^2}\sin^2\dfrac{A}{2}$.

If b and c be nearly equal to each other, show why θ cannot be determined with accuracy by the aid of logarithmic tables; and explain how you would proceed in that case.

4. Find an expression for the radius (R) of the circle circumscribing a triangle ABC. If O be the centre of this circle, and R_1, R_2, R_3 be the radii of the circles which circumscribe respectively BOC, COA, AOB, prove that $R_1\cos A=R_2\cos B=R_3\cos C$, and that

$$\frac{1}{R_1^2}+\frac{1}{R_2^2}+\frac{1}{R_3^2}+\frac{R}{R_1R_2R_3}=\frac{4}{R^2}.$$

* 5. Enunciate De Moivre's Theorem, and deduce the equation—

$$\cos\theta=1-\frac{\theta^2}{\underline{|2}}+\frac{\theta^4}{\underline{|4}}-\frac{\theta^6}{\underline{|6}}+\ \dots\ .$$

6. Find the sum of the series—

(i.) $\sin\theta+\sin 2\theta+\sin 3\theta+\ \dots\ $ to n terms.

(ii.) $\sin\theta-\dfrac{\sin 2\theta}{\underline{|2}}+\dfrac{\sin 3\theta}{\underline{|3}}-\ \dots\ $ to infinity.

C.

1. Enunciate and prove the logarithmic theorem.

Find the first three terms of the expansion of $\log\tan x-\log x$ in powers of x.

2. Prove the formula for $\cos(A-B)$, using a figure in which $A=320°$, and $B=200°$.

If the sum of three angles is zero, and the sum of their cosines is -1, prove that the product of their cosines is equal to the square of one of them.

3. Find $\sin\dfrac{A}{2}$ in terms of $\sin A$, and prove that while the radicals retain the same signs, A must lie between limits differing by π.

If the product of the tangents of three acute angles be unity, find the minimum value of the product of their sines.

4. Explain the ambiguous case in the solution of plane triangles, and solve the triangle in which $a=120$, $b=90$, $B=50°$.

Give a trigonometrical solution of the following problem :—Construct a square with its sides passing respectively through the angular points of a given convex quadrilateral. Show when the solution is impossible.

5. Prove that the radii of the four circles touching the sides of a triangle, that of the inscribed circle, being taken with an opposite sign to the rest, satisfy the equation $r^4+ar^3+\beta r^2+\gamma=0$.

Find the sides and the area of a triangle in terms of the radii of the escribed circles.

6. Sum to n terms the series $\sin\theta+2\sin 2\theta+3\sin 3\theta+$ etc.; and to infinity the series whose general terms is

$$\{1.3\ \ldots\ .\ (2n-1)\}^2\frac{\cos(2n+1)\theta}{\underline{|2n+1}}.$$

CI.

* 1. Find an expression for all the angles having a given tangent.

If $\tan(\pi\cot\theta)=\cot(\pi\tan\theta)$, find all the values of θ.

* 2. Prove that any formula which is true for the angles A, B, C of a triangle, will be true when A, B, C are replaced respectively by $\frac{\pi-A}{2}$, $\frac{\pi-B}{2}$, $\frac{\pi-C}{2}$, or by $\pi-2A$, $\pi-2B$, $\pi-2C$.

* Prove that in any triangle—

$$\sin\left(A+\frac{B}{2}\right)+\sin\left(B+\frac{C}{2}\right)+\sin\left(C+\frac{A}{2}\right)+1$$
$$=4\cos\frac{B-C}{4}\cos\frac{C-A}{4}\cos\frac{A-B}{4}.$$

3. Solve the triangle in which two sides are respectively 89 and 130 feet, and the angle opposite the former 35° 18′ 29″; also the triangle whose sides are 1, $1+x$, $1+y$, where x and y are very small quantities.

4. Find expressions for the radii of the inscribed and circumscribed circles of a triangle.

If r_1, and r_2 be the radii of the greatest and least circles which can be inscribed between the inscribed and circumscribed circles, and r the radius of the inscribed circle, prove that $4r_1 r_2=r^2$.

5. Investigate the formulæ $\cos\theta \pm \sqrt{-1}\sin\theta = e^{\pm\theta\sqrt{-1}}$.

Express $\frac{1}{\sqrt{-1}}\sin^{-1}\frac{x\sqrt{-1}}{a}$ in a rational form.

6. If $a = 1 + z\cos\theta + \frac{z^2}{\underline{|2}}\cos 2\theta + \frac{z^3}{\underline{|3}}\cos 3\theta + \ldots$

and $b = z\sin\theta + \frac{z^2}{\underline{|2}}\sin 2\theta + \frac{z^3}{\underline{|3}}\sin 3\theta + \ldots.$

find $z\sin\theta$ and $z\cos\theta$ in terms of a and b.

* 7. Resolve $\cos\theta$ into factors.

CII.

1. Find an expression for all angles having the same tangent.

Show whether the trigonometrical solution of the triangle ABC is ambiguous or not, where $BAC = 30°$, $AC = 5$, $BC = 4$.

2. Give the sine of 30°, explain the steps of the process for calculating the sine of one degree. What is herein assumed as the unit of angle, and at what stage of the process?

3. Reduce $A\sin\theta + B\cos\theta$ to a form adapted to logarithmic computation.

4. Of what number is 5 the logarithm to the base 4? Show how a table of logarithms might be calculated by the continual extraction of the square root.

5. Prove that $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$, and from thence obtain $\pi = 3{\cdot}1416 \ldots.$ State some more elementary method of finding the ratio of the length of the circumference of a circle to that of its diameter.

6. An engineer, outside a town, is provided with a map of the buildings of the interior; what angles must he measure so as to determine his distance from any three of them? and explain the method.

* 7. If $\tan\theta = \frac{b}{a}$,

$$\text{then } (a \pm b\sqrt{-1})^{\frac{1}{n}} = (a^2 + b^2)^{\frac{1}{2n}}\left\{\cos\frac{\theta}{n} \pm \sqrt{-1}\sin\frac{\theta}{n}\right\}.$$

CIII.

1. Prove the following—

(i.) $\sin 30^\circ = \cos 60^\circ = \cdot 5.$

(ii.) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}.$

(iii.) $\cot^{-1} a + \cot^{-1}(4a^3 + 3a) = 2 \cot^{-1} 2a.$

and if $\tan\dfrac{\beta - a}{2} = \dfrac{3 \sin a}{5 - \cos a}$, then $\tan\dfrac{\beta}{2} = 4 \tan\dfrac{a}{2}$.

2. If A, B, C are the angles of a plane triangle, and a, b, c are the opposite sides, and R is the radius of the circumscribed circle, prove that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R} = \frac{2}{abc}\left\{ s(s-a)(s-b)(s-c) \right\}^{\frac{1}{2}}.$$

Also find in terms of the sides of the triangle the area of the triangle whose angular points are the centres of the three escribed circles.

3. Prove that $\log_{10} n = \dfrac{1}{\log_e 10} \times \log_e n$; and show how $\log_e 10$ is found.

4. Prove formulæ adapted to logarithmic computation for the solution of a plane triangle, when two sides and the included angle are given.

If two sides of a triangular field are 470·61 and 893·4 feet respectively, and the angle contained between them is 58° 20′, find the length of the other side, and the area of the field in acres.

5. Assuming De Moivre's Theorem to be true for positive integral indices, prove it to be true for negative indices.

Express $\tan nx$ in terms of powers of $\tan x$.

6. Sum the following series—

(i.) $\sin\theta + 2\sin 2\theta + 3\sin 3\theta + \ldots$ to n terms.

(ii.) $\cos\theta + \dfrac{\cos 2\theta}{2} + \dfrac{\cos 3\theta}{3} + \ldots\ldots$ to infinity.

CIV.

1. Prove that $\cos(A+B) = \cos A \cos B - \sin A \sin B$, and deduce the formula for $\sin(A-B)$.

2. Find an expression for the area of a plane triangle in terms of (1) its three sides, (2) two of its sides and the included angle.

ABC, $A'B'C'$, are two acute-angled triangles; show that if $BC > B'C'$, $CA > C'A'$, $AB > A'B'$, then the triangle ABC is greater than the triangle $A'B'C'$. Would this assertion be necessarily true if one or both of the triangles were obtuse-angled?

3. If a, b, c, A, B, C, are the sides and angles of a given triangle, find the angles of the triangle of which the sides are—

$$\frac{a}{\sin\frac{A}{2}},\quad \frac{b}{\sin\frac{B}{2}},\quad \frac{c}{\sin\frac{C}{2}}.$$

4. The angles of elevation of the top of a tower are observed at two stations, of which the distance is known, and which are in the same horizontal plane with the base of the tower. The angle subtended by the distance between the two stations is also observed from the top of the tower. Determine its height.

5. Prove the series for the expansion of e^x, and show that—

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x.$$

CV.

1. Assuming the ordinary approximate value of π, calculate, to the nearest integer, the number of seconds in the angular unit of circular measure.

* 2. Prove, by any method, the formulæ for $\sin(A+B)$, and for $\cos(A+B)$ in terms of the sines and cosines of A and B.

* 3. Assuming the preceding formula for $\sin(A+B)$, prove those for $\sin A$ and $\cos A$ in terms of $\sin 2A$.

4. Given, of a plane triangle, a side c, and the two adjacent angles A and B; determine, by trigonometry, the remaining sides a and b.

5. Given, of a plane triangle, an angle C, and the two adjacent sides a and b, determine, by trigonometry, the remaining angles A and B.

6. Express in each of the preceding cases, the length of the perpendicular p from the vertex C upon the side c, in terms of the quantities given in each case.

7. Given that, $\cos x + \cos y = a$, $\sin x + \sin y = b$; find the values of x and y in terms of a and b.

8. Given that $(a \pm b\sqrt{-1})^{\frac{1}{2}} = x \pm y\sqrt{-1}$; find the values of x and y in terms of a and b.

* 9. Investigate, by any method, the series for $\log_e (1+x)$ in ascending powers of x.

10. If n be any large number, and d any small increase or diminution to it, prove the approximate formula—

$$\log_a (n \pm d) = \log_a n \pm \frac{1}{\log_e a} \cdot \frac{d}{n}.$$

* 11. Prove De Moivre's Theorem for positive integral indices; and hence express (1) $\cos n\theta$ in terms of $\cos \theta$ and its powers; and (2) $(\cos \theta)^n$ in terms of cosines of the multiple arcs.

CVI.

1. Investigate the formula—

$$\sin (A+B) = \sin A \cos B + \cos A \sin B.$$

* Deduce the formula for $\cos (A+B)$.

Solve the equation $3 \tan 3x - \tan 2x = \tan x$.

2. Assuming the expression for the cosine of an angle of a triangle in terms of the sides, find the expression for the tangent of half an angle.

* If the tangents of the half angles of a triangle are in harmonical progression, show that the sides of the triangle are in arithmetical progression.

3. Show how to solve a triangle, having given two sides and the included angle.

If $a=210$, $b=110$, $C=34° \, 42' \, 30''$, find the other angles, having given

$$\log 2 = \cdot 3010300, \quad L \cot 17° \, 21' \, 15'' = 10 \cdot 5051500.$$

4. Investigate expressions for the radii of the inscribed and the circumscribed circles of a triangle.

Show that the former radius cannot be greater than half the latter.

CVII.

1. Show that $\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$.

If $A + B + C = 2\pi$, prove that—

$$\cos A + \cos B - \cos C = 1 - 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$$

* 2. Investigate the expansion of x^n in powers of n.

Apply your result to show that the series—

$$1 + 2 + \frac{3}{1.2} + \frac{4}{1.2.3} + \ldots = 2e.$$

3. Show how you would construct a table of the sines and cosines of all angles differing by one minute, from 0° to 90°.

* 4. If A and B are two inaccessible points in a given plane, how can the distance between them be determined by trigonometry?

* 5. Investigate the exponential values of $\sin x$ and $\cos x$.

Apply them to sum the series $\cos a + x \cos (a + \beta) + x^2 \cos (a + 2\beta) + \ldots$ to infinity, where $x < 1$.

CVIII.

* 1. Given $\cos A$, find $\cos 3A$. Show how the equation so found is made auxiliary to the solution of a cubic equation.

2. Explain fully the ambiguous case in the solution of a plane triangle, and show, geometrically, that the trigonometric double value coincides with the answer found by construction.

* 3. Prove De Moivre's Theorem—

$$\cos \frac{m\theta}{n} \pm \sqrt{-1} \sin \frac{m\theta}{n} = (\cos \theta \pm \sqrt{-1} \sin \theta)^{\frac{m}{n}}.$$

Show that there are the same number of values of the expression on each side of this equation.

* 4. Prove that $\tan \theta - \frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta - \ldots = \theta$; and apply the series to calculate the ratio of the circumference to the diameter of a circle.

CIX.

1. Define the circular measure of an angle in Trigonometry, and show how to reduce to degrees, minutes, and seconds an angle given in circular measure, and conversely. Reduce to degrees, minutes, and seconds the angle whose circular measure $= \frac{1}{2}$.

2. Assuming the formula for $\sin (A+B)$ and for $\cos (A+B)$, investigate those for $\sin 3A$ and for $\cos 3A$ in terms of $\sin A$ and of $\cos A$ respectively. Apply the latter formulæ to determine, in their surd forms, the numerical values of $\sin 18^\circ$ and of $\cos 18^\circ$.

3. Investigate a series for the expansion of a^x in ascending powers of x; and show how the coefficient of the first power of x is determined. If e be the value of a for which the coefficient $=1$, give the series expressing its numerical value.

4. Given $\log_e n$, investigate the series for $\log_e (n+1)$ in ascending powers of $(2n+1)^{-1}$, and explain its convenience when n is a large number. Calculate the numerical value of $\log_e 2$ to five places of decimals.

* 5. Investigate, by any method, the formula for the tangent of the sum of any number of angles in terms of the tangents of the several angles. The tangents of the three angles are respectively $\frac{1}{2}$, 1, 2; calculate the tangent of their sum.

6. Investigate, by any method, the logarithmic formula for the area of a plane triangle in terms of the three sides.

The three sides of a plane triangle are 13, 14, 15; calculate its area.

CX.

1. Prove the formula, $\sin (A+B) = \sin A \cos B + \cos A \sin B$.

If $\sin A = \frac{5}{13}$, and $\sin B = \frac{3}{5}$, find $\cos (A+B)$.

2. Prove that—

$$\sin A\,(1+\tan A) + \cos A\,(1+\cot A) = \sec A + \operatorname{cosec} A.$$

3. Investigate the expression for the cosine of an angle of a triangle in terms of the sides.

Lines are drawn from the angular points of a triangle to the middle points of the opposite sides; if p, q, r denote the lengths of these lines, show that—

$$p^2+q^2+r^2=\frac{3}{4}\ (a^2+b^2+c^2)$$

4. Show how to solve a triangle when two sides and the included angle are given.

Example.—Suppose the sides to be 9 and 18, and the included angle 60°.

5. Find an expression for the area of a triangle in terms of the sides. If the triangle be right-angled, show that the expression reduces to half the product of the sides which contain the right angle.

* 6. Find the sum of n terms of the series—

$$\sin a+\sin(a+\beta)+\sin(a+2\beta)+\ .\ .\ .$$

Deduce the sum of n terms of the series—

$$\cos a+\cos 2a+\cos 3a+\ .\ .\ .$$

CXI.

1. What are the base and characteristic of a logarithm? What is the characteristic of the logarithm of ·003 to base 5, and that of $\frac{1}{31}$ to base 3?

2. Give the geometrical definitions of the tangent and secant of an angle; and trace each of them through all their changes of sign and magnitude.

3. Prove that $2(\sin^2 A\ \sin^2 B+\cos^2 A\ \cos^2 B)=1+2\cos 2A\ \cos 2B$.

4. At three given stations along any horizontal line not meeting the vertical line from a balloon, the height of which it is wished to determine, three persons observe simultaneously its angles of elevation a, β, and γ, from their respective stations; show how to determine the height of the balloon.

CXII.

* 1. Explain the ambiguous case in the solution of plane triangles.

* 2. Expand the expression $\log_e \tan(\theta + n'') - \log_e \tan\theta$ to small quantities of the second order. Hence show that the tables of proportional parts affixed to the common logarithmic tables will not be applicable when it is required to determine accurately an angle which is either very small or nearly a right angle, from its logarithmic tangents, or the converse.

3. Show how three observers at three known stations on the same straight line on a horizontal plane may obtain the height of a cloud by three simultaneous observations of its angular elevation.

CXIII.

1. Explain the principle of the method by which the sines of very small angles are calculated; and apply it, assuming the value of π, to find the numerical value of the sine of $5''$.

2. State and prove the most rapid method with which you are acquainted of approximating to the value of π, and apply it to determine its value to the five places usually given.

3. A tower of known height a stands on a hill of unknown height, rising from a horizontal plane. Show how an observer, with the means of observing angles in a vertical plane, may determine the height of the hill by observations from a point C situated at an unknown distance on the horizontal plane.

4. State and prove the ordinary rule by which the angle corresponding to a logarithmic tangent or cotangent not given in the tables is calculated, and show when it becomes inapplicable.

CXIV.

1. Define the terms "logarithm" and "base of a system of logarithms." Enunciate and prove those properties of logarithms which render them so useful in calculation.

* Given $\log_{10} 5 = \cdot 699$, find the number of places of figures in 2^{63}.

* 2. Deduce from a figure the general formula for all arcs of the unit circle which have (1) the same cosine, (2) the same cotangent.

3. Prove the following—

$$\text{(i.)}\ \tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}.$$

$$\text{(ii.)}\ \sin^{-1}\frac{x-y}{x+y} = \tan^{-1}\frac{x-y}{2\sqrt{xy}}.$$

$$\text{(iii.)}\ \tan^{-1}\frac{\cos a}{x-\sin a}\ \tan^{-1}\frac{1-x\sin a}{x\cos a} = a.$$

and eliminate θ from $\dfrac{a-b}{a+b} = \dfrac{\tan(\theta-a)}{\tan(\theta+a)}$ and $a\cos 2a + b\cos 2\theta = c$.

4. The angle of elevation of the top of a tower is observed at three known stations in a horizontal plane; show that these data are sufficient to determine the position and height of the tower, and investigate the requisite formulæ.

* 5. Show by a rigorous method that

$$a^x = 1 + x\log a + \frac{x^2}{\underline{|2}}(\log a)^2 + \ldots$$

What is the base of the logarithm in this case, and how may its numerical value be calculated?

6. Prove the truth of the formula $\sin\theta = \theta - \dfrac{\theta^3}{\underline{|3}} + \dfrac{\theta^5}{\underline{|5}} - \ldots$ showing why θ must be expressed in circular measure, and pointing out within what limits it must lie.

CXV.

1. If a be the arc of a circle subtending an angle θ, and r the radius of the circle, show that $\theta = \frac{180^\circ}{\pi} \cdot \frac{a}{r}$, where $\pi = 3{\cdot}14159$.

What will this become if we take an angle equal to one-sixth of a right angle as the angular unit?

2. Express $\operatorname{vers}\theta$ in terms of $\operatorname{cosec}\theta$, and hence determine the numerical values of $\operatorname{vers}\theta$ for $\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \pi$.

If $\tan^2\theta = 3$, express all the positive values of θ which satisfy the equation.

3. What is meant by the logarithm of a number to a given base? and its characteristic? What is the characteristic of the logarithms of 35 and of ·0023 respectively to base 3? What are the advantages of a system of logarithms to base 10?

4. Prove the formula $\log_a(1+x) = \frac{1}{\log_a e}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right)$.

5. The plane side of a hill running E. to W. is inclined to the horizon at an angle α; it is required to construct a straight railroad upon it inclined at an angle β to the horizon: determine the point of the compass to which it must be directed.

6. A tower situated on a horizontal plane subtends an angle of 2° at a station A, and at another station B, 100 yards nearer the tower, it subtends an angle of 3°, A, B, and the foot of the tower being in the same straight line; determine approximately the height of the tower in feet.

7. Assuming De Moivre's Theorem, write down the fourth roots of $+1$, of -1, and of $+\sqrt{-1}$.

CXVI.

1. Prove that $\sin(A+B)=\sin A\cos B+\cos A\sin B$; for the particular case in which A is between one and two right angles, B less than a right angle, and $A+B$ greater than two right angles.

(The proof is to be independent of that for the case in which all the angles are acute.) Hence show the use of the negative sign in generalising the above formula for all values of the angles.

2. Find all the positive values of θ which satisfy the equation $\sin\theta=-\cos a$. Find all those which satisfy the equation—

$$\tan\frac{\theta}{2}+\frac{1}{4}\operatorname{cosec}\theta-\sin\theta=0.$$

3. Eliminate c between the equations $a^2=b^2+c^2-2bc\cos A$, and $b^2=a^2+c^2-2ac\cos B$.

4. Assuming that $\log_e(1+x)=\frac{1}{\log_e 10}\left\{x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right\}$, investigate a convenient form for the calculation of the logarithms of consecutive prime numbers.

Prove that $\log_{10} N=\frac{m}{\log_e 10}\left\{(N^{\frac{1}{m}}-1)-\frac{1}{2}(N^{\frac{1}{m}}-1)^2+\ldots\right\}$ and explain how this formula has been used for the calculation of $\log_e 10$ to a great number of places of decimals.

* 5. Prove, independently of differentiation, that

$$\log_a\sin(\theta+n'')-\log_a\sin\theta=\frac{1}{\log_e a}\left\{\cot\theta\cdot h-\operatorname{cosec}^2\theta\,\frac{h^2}{|\underline{2}}+\ldots\right\}$$

where $h=\frac{\pi n}{180\times 60\times 60}$.

Explain how by this formula we may construct the tables of proportional parts attached to the ordinary tables of logarithmic sines, tangents, etc.

* 6. Express $\sqrt[6]{-1}$ in the form $\alpha+\beta\sqrt{-1}$; determine all the values of α and β.

CXVII.

1. Find by logarithms the number of digits in $(512)^5$; and find also by logarithms a fourth proportional to ·00625, ·005, ·064, having given $\log_{10} 2 = \cdot 301030$.

2. If $\cos \alpha = \cos \beta \cos \theta$, prove $\tan \frac{\alpha + \beta}{2} \tan \frac{\alpha - \beta}{2} = \tan \frac{\theta}{2}$.

3. From the simultaneous equations—

$$x \cos \alpha + y \sin \alpha = p, \quad x \cos \alpha_1 + y \sin \alpha_1 = p, \quad x \cos \alpha_2 + y \sin \alpha_2 = p_2,$$

obtain the relation $\frac{\sin (\alpha_1 - \alpha_2)}{p_1 p_2} \quad \frac{\sin (\alpha_2 - \alpha)}{p p_2} \quad \frac{\sin (\alpha - \alpha_1)}{p p_1} = 0.$

4. Find the area of a quadrilateral figure inscribed in a circle, in terms of the sides.

If a, b, c, d be the four sides of any quadrilateral figure, d_1, d_2 the diagonals, e the straight line joining the middle points of the diagonals, prove $a^2 + b^2 + c^2 + d^2 = 4e^2 + d_1^2 + d^2_2$.

CXVIII.

1. Deduce the number which gives the ratio of the circumference of a circle to its diameter, namely 3·14159.

* 2. Eliminate concisely x and y between the equations—

$$\tan x + \tan y = a, \quad \cot x + \cot y = b, \quad x + y = c.$$

* 3. The following rule is given for determining the distance of the sea horizon in miles: "To the height of the eye in feet add half the height, and extract the square root of the sum." Account for this.

CXIX.

* 1. Eliminate a, b, c from the three equations $b \cos C + c \cos B = a$, $c \cos A + a \cos C = b$, $a \cos B + b \cos A = c$, which subsist between the sides a, b, c and the angles A, B, C of a plane triangle. Show the geometrical meaning of the resulting equation.

* 2. Prove the following formulæ—

$$\sin \frac{A - B}{2} = \frac{a - b}{c} \cos \frac{C}{2}, \quad \cos \frac{A - B}{2} = \frac{a + b}{c} \sin \frac{C}{2}$$

by means of geometrical constructions.

* 3. Prove the following theorem—

$$\log \left\{ (1 + N)^{\frac{1+N}{2}} \cdot (1 - N)^{\frac{1-N}{2}} \right\} = \frac{N^2}{1.2} + \frac{N^4}{3.4} + \frac{N^6}{5.6} + \dots$$

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CXX.

1. Prove that $1+\tan A \tan \frac{A}{2} = \tan A \cot \frac{A}{2} - 1 = \sec A$.

* Find all the values of θ which satisfy the equation—

$$\sin \frac{n+1}{2}\theta + \sin \frac{n-1}{2}\theta = \sin \theta.$$

2. Two straight lines, OA, OB, of length a and b respectively, are inclined to each other at an angle α, and straight lines AP, BP, meeting in P, are drawn at right angles to OA and OB; prove that if the point P fall within the angle α, the area of the quadrilateral $OAPB$ is

$$\frac{2ab-(a^2+b^2)\cos \alpha}{2\sin \alpha}.$$

3. The sides of a triangle are 25, 26, 27; find the angles, having given $\log 3 = {\cdot}4771213$, $L\tan 31^\circ 56' 50'' = 9{\cdot}7948986$, $L\tan 31^\circ 57' = 9{\cdot}7949455$.

4. Sum to n terms the series of which the nth term is—

$$\frac{2^{n-1}\sin 2^{n-1}\theta}{2\cos 2^{n-1}\theta - 1};$$

and to infinity the series—

$$1 - \frac{n(n+1)}{1.2}\tan^2 \frac{\pi}{2^n} + \frac{n(n+1)\ldots(n+3)}{\underline{|4}}\tan^4 \frac{\pi}{2^n} + \ldots$$

* $1 + 2a\cos\theta + 3a^2\cos 2\theta + 4a^3\cos 3\theta + \ldots$

CXXI.

1. Obtain an expression for all the angles having the same sine as α.

Find the most general value of θ satisfying both the equations

$$\sin\theta=\frac{1}{2},\quad \tan^2\theta=\frac{1}{3}.$$

2. Prove that—

(i.) $\left\{1+\tan\frac{\alpha}{2}+\sec\frac{\alpha}{2}\right\}\left\{1+\tan\frac{\alpha}{2}-\sec\frac{\alpha}{2}\right\}=\sin\frac{\alpha}{2}\sec^2\frac{\alpha}{2}.$

(ii.) $2\tan^{-1}\left\{\tan(45^\circ-\alpha)\tan\frac{\beta}{2}\right\}=\cos^{-1}\left\{\frac{\tan\alpha+\cos\beta}{1+\tan\alpha\cos\beta}\right\}.$

3. If three straight lines be drawn from a point to the three angular points of an equilateral triangle, find an equation to determine the length of a side of the triangle in terms of the lengths of these lines.

* 4. Find the angles of a triangle, where $3c=7b$, and $A=6^\circ\ 37'24''$, having given

$\log 2 = \cdot 301030$ $\qquad$ $L\tan 8^\circ\ 13'\ 50''=9\cdot 1603083$

$L\tan 3^\circ\ 18'\ 42''=8\cdot 7624069$ $\qquad$ diff. for $10''=1486.$

5. From an examination of the factors of $x^{2n}-2x^n\cos 2\theta+1$, deduce the expression for $\sin\theta$ in factors, viz.—

$$\sin\theta=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\cdot\cdot\cdot\cdot$$

Show that $\operatorname{cosec}\theta=\frac{1}{\theta}-\Sigma_1^\infty\left\{\frac{4\theta}{4n^2\pi^2-\theta^2}-\frac{2\theta}{n^2\pi^2-\theta^2}\right\}.$

6. Find the sum of the following series—

(i.) $\sin\theta\sin 2\theta\sin 3\theta+\sin 2\theta\sin 3\theta\sin 5\theta+\sin 3\theta\sin 4\theta\sin 7\theta+\ldots$ etc. to n terms.

(ii.) $c\sin\theta+\frac{c^2}{1.2}\sin 2\theta+\frac{c^3}{1.2.3}\sin 3\theta+$ etc. to infinity.

CXXII.

1. Prove the following formulæ—

(i.) $\tan(30^\circ + A)\tan(30^\circ - A) = \dfrac{2\cos 2A - 1}{2\cos^2 A + 1}$.

(ii.) $\sin\alpha + \sin\beta + \cos\alpha + \cos\beta = \pm 2\cos\dfrac{\alpha - \beta}{2}\sqrt{1 + \sin(\alpha + \beta)}$.

2. In order to calculate the height of a cliff, an observation is taken from a fixed position, and the angular altitude is found to be 12° 30′; a second observation is taken from a point 950 yards nearer to the cliff, and the angular altitude is found to be 69° 30′: find the height of the cliff, and the distance of the first station from its base.

Given log 2 = ·30103 log 19 = 1·27875
L sin 12° 30′ = 9·33534 L sin 57° = 9·92359
L sin 20° 30′ = 9·54433 L cos 20° 30′ = 9·97159
log ·8586 = ·93379 log ·2296 = ·36097
log ·2297 = ·36116.

3. Prove that the radius of each of the two escribed circles of an isosceles triangle is equal to the altitude of the triangle.

4. In a given circle an equilateral triangle is inscribed, and in that triangle another circle, and so on *ad infinitum;* show that the sum of the circumferences of all the inner circles is equal to that of the given circle, and the sum of their areas to a third of its area.

*5. Prove the expansion of a^x in a series of ascending powers of x, and hence deduce the value of e, and prove that e is an incommensurable quantity.

*6. State De Moivre's Theorem, and prove that when the index is fractional, and of the form $\dfrac{p}{q}$, q and no more than q different values of the expansion can be obtained.

7. Prove a formula for the expansion of $\sin\alpha$ in terms of ascending powers of α, and show that the series is convergent.

8. By means of the exponential expressions for $\sin\theta$ and $\cos\theta$, prove the following formulæ—

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}, \quad \sin 2\theta = 2\sin\theta\cos\theta;$$

and sum the series—

$$\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots \text{ etc., to } n \text{ terms.}$$

CXXIII.

1. Trace the sign of $\sin 2\theta - \sin\theta$, as θ increases from 0° to 180°.

2. Prove that—

$$\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}+\frac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}=2\operatorname{cosec}\theta;$$

and if $\cos 2A + \cos(n-2)A = \cos A$, find the general value of A.

3. Two landmarks, exactly north and south of each other, are separated by a river. A person wishing to know the distance between them walks due west from one, and measures the angle subtended at the eye by the two objects, he then walks double the distance in the same direction, and finds that the subtended angle is half what it was before. Determine the required distance.

4. Having given two sides, and the included angle of a triangle, solve the triangle. How many logarithms are required for the complete solution?

If $b=5$, $c=3$, $A=120°$, find the other angles, having given—

$\log 4{\cdot}8 = {\cdot}6812412$ $\quad$ $L \tan 8° 12' = 9{\cdot}1586706$

diff. for $60'' = {\cdot}0008940$.

5. Having given the four sides of a quadrilateral figure inscribed in a circle, find its area and angles, and the radius of the circle.

* 6. Assuming De Moivre's Theorem, deduce the expansion of $\sin n\theta$, and $\cos n\theta$, in terms of powers of $\sin\theta$ and $\cos\theta$, and write down the last term in both series.

7. Expand θ in terms of $\tan\theta$, and show how the value of π may be calculated. What is meant by the impossibility of squaring the circle?

* 8. Find the coefficient of x^n in the expansion of $e^x . \cos x$.

* 9. Investigate a series by which, when the logarithm of a number is given, the logarithms of the two consecutive numbers may be correctly found.

CXXIV.

1. Prove that if $\sin A$ be given, $\sin \frac{A}{2}$ has in general four values. Find in what cases these four values reduce to three or to two.

* 2. Prove the following formulæ in a triangle—
(i.) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.
(ii.) $\sin 3A \sin (B - C) + \sin 3B \sin (C - A) + \sin 3C \sin (A - B) = 0$.

3. Find $\tan (A + B + C)$ in terms of $\tan A$, $\tan B$, $\tan C$. Solve the equation $n \tan (\theta - a) + m \tan (\theta + a) = (n - m) \cot a$.

4. Solve the triangle in which $a = 75$, $b = 40$, and $B = 69° 18'$.

5. Prove that if R, r, be the radii of the circumscribed and inscribed circles of a triangle, and r_1, r_2, r_3 the radii of the escribed circles, then $r_1 + r_2 + r_3 - r = 4 R$, and $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$.

* 6. Prove every number has an infinite number of imaginary logarithms.

7. Show that $\tan^{-1} x = n\pi + x - \frac{x^3}{3} + \frac{x^5}{5} - \text{etc.}$
Find the quantity of which $\sqrt{-1} \tan^{-1} x\sqrt{-1}$ is the logarithm.

8. Investigate the formula for $\cos^n x$ in cosines of multiple angles. Prove that $\cos mx = 2^{m-1} \cos^m x +$ lower powers of $\cos x$.

CXXV.

1. Define the tangent of an angle, and prove that—

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

If $A + B + C = 180°$, prove that—

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$
$$\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C.$$

2. Find the sine and cosine of 18°. Determine the values of θ, between 0 and $\frac{\pi}{2}$, which satisfy the equation $\sin^2\theta + \cos^2 2\theta = \frac{3}{4}$.

3. Show how to solve a triangle when two sides and the included angle are given.

If $b = 21$, $c = 11$, $A = 34° \; 42' \; 30''$, find the other angles, having given
$\log 2 = \cdot301030$ $\quad$ $L \cot 17° \; 21' \; 15'' = 10\cdot50515$.

4. Find the sum of the series—

$\sin\theta + \sin 2\theta + \sin 3\theta + \; . \; . \; .$ to n terms.

$\sin\theta + \sin\theta \sin 2\theta + \sin^2\theta \sin 3\theta + \sin^3\theta \sin 4\theta + \; . \; . \; .$ to infinity.

CXXVI.

1. Write down the formula for all angles having the same sine, and find the general solution of the simultaneous equations—

$$\left.\begin{aligned} \sin(2x + 3y) &= \cos a \\ 3x + 2y &= \pi. \end{aligned}\right\}$$

* 2. Given $A = 55°$, $B = 65°$, and $c = 270$ yards, find a, having given

$L \sin 55° = 9\cdot9133645$ $\quad$ $\log 2\cdot5553 = \cdot4070508$
$\log 3 = \cdot4771213$ $\quad$ $\log 2\cdot5554 = \cdot4072209$.
$\log 2 = \cdot3010300$

3. Find the radius of the circle inscribed in a given triangle, and prove that it cannot be greater than half that of the circumscribing circle.

* 4. Write down the series for $\cos\theta$ in ascending powers of θ, and deduce the exponential expression for $\cos\theta$.

5. Find the sum of the infinite series whose rth term is $\frac{x^r \sin ra}{\lfloor r}$.

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(Obligatory.)

CXXVII.

1. Find the number of degrees, minutes, and seconds in the angle $34^{g}\, 68^{\backprime}\, 54^{\backprime\backprime}$; also, find the circular measure of the complement of this angle. $\left(\text{Given } \pi = \frac{22}{7}.\right)$

* 2. Find by geometrical construction the sine and the cosine of the sum of two angles, each of which is less than a right angle and their sum greater than a right angle, in terms of the sines and cosines of the angles themselves.

* 3. Prove, geometrically, that—

$$\sin 2A = 2 \sin A \cos A; \text{ and } \sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}.$$

4. Prove the formulæ—

$$\text{(i.) } \cos^2 (A - B) - \sin^2 (A + B) = \cos 2A \cos 2B.$$

$$\text{(ii.) } \sec (A \pm B) = \frac{\sec A \sec B}{1 \mp \tan A \tan B}.$$

$$\text{(iii.) } \cot \frac{A}{2} = \frac{\sin A + \sin 2A}{\cos A - \cos 2A}.$$

5. Find sin 9° and cos 36°; and show that—

$$\sin^2 54^\circ - \sin^2 18^\circ = \frac{\sqrt{5}}{4}.$$

6. If in any triangle *ABC*, two sides and the angle they include be given, state and prove the proposition, geometrically or analytically, by means of which the other angles may be determined.

Given $b = a(\sqrt{3} - 1)$, and $C = 30^\circ$; find the angles A and B.

* 7. In the ambiguous case in the solution of triangles if A, B, C, A, B', C' be the angles of the triangles in the two solutions; prove that—

$$\frac{\sin C}{\sin B} + \frac{\sin C'}{\sin B'} = 2 \cos A.$$

8. The sides of a triangle are respectively 24, 31, and 11 feet; determine the greatest angle.

9. From a ship at sea it is observed that the angle between two forts A and B is 30°; the ship sails for 4 miles towards A, and the angle between the forts is then observed to be 48°; prove that the distance of the ship from B at the second observation is 6·472 miles.

10. In the middle of a field which is an equilateral triangle there stands a tower 200 feet high. From the top of the tower each side of the field is observed to subtend an angle $\cos^{-1}\frac{1}{10}$. Find a side of the field.

* 11. The sides of a triangle are in arithmetical progression, and its area is to that of an equilateral triangle of the same perimeter as 3 : 5. Find the ratio of the sides and the value of the largest angle.

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CXXVIII.

* 1. Explain the different methods of measuring angles in Trigonometry. If D, G, and C be respectively the number of degrees, grades, and units of circular measure in an angle, show that—

$$\frac{D}{90}=\frac{G}{100}=\frac{2C}{\pi}.$$

* 2. Prove that—

(i.) $\sin(A+B)=\sin A\cos A+\cos A\sin B$, and thence find sin 75°.

(ii.) $2\sec A=\tan\left(45^\circ+\frac{A}{2}\right)+\cot\left(45^\circ+\frac{A}{2}\right)$.

(iii.) $\frac{\cos A+\sin A}{\cos A-\sin A}=\tan 2A+\sec 2A$.

3. Find sin 45° and cos 36°. Also L sin 45°. Also all the values of θ which satisfy the equation $\tan\theta+\cot\theta=\frac{4}{\sqrt{3}}$.

4. Having given that $2\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}$, show how to determine what signs must be used before the radicals when A lies between 450° and 540°.

5. In a plane triangle of which A, B, C are the angles, and a, b, c the opposite sides, show that—

(i.) $a \sin B = b \sin A$.

(ii.) $(a+b) \cos \frac{A+B}{2} = c \cos \frac{A-B}{2}$.

(iii.) $a \cos (B-C) = b \cos B - c \cos C$.

6. Given one side of a triangle, one angle adjacent to it, and the area of the triangle, solve the triangle. A field has one side 220 yards long, and the adjacent side makes an angle of 72° with it; find the point in the latter side from which a straight line must be drawn to the farther extremity of the former side so as to cut off exactly one acre.

7. In a plane triangle, having given two sides and the included angle, obtain the formulæ for solving the triangle.

Ex. Given— $a=205$, $b=195$, $C=4°$,

$\log_{10}2 = \cdot30103$, $L \cot 2° = 11\cdot4569162$,

$L \cot 54° \, 20' = 9\cdot8559376$, $L \cot 54° \, 21' = 9\cdot8556708$,

find the remaining angles; and explain why in general, when $a=b$ nearly, the usual solution is not trustworthy, and why it may be employed in the case proposed.

8. A man observes two objects in the same plane with his eye to be 30° apart, and on walking 50 yards in a straight line towards one of them the angle increases to 75°; find his distance from the other. Show whether the data are sufficient to find the distance between the objects.

*9. Show that the difference of the logarithms of the sines of two angles is nearly proportional to the difference of the angles, when that difference is small.

Some of the following logarithms will be required.

$\log 1\cdot1 = \cdot0413927$.

$\log 2 = \cdot3010300$.

$\log 3 = \cdot4771213$.

$\log 3\cdot14159 = \cdot4971495$.

$\log 3\cdot4377 = \cdot5362680$, diff. 126.

$\log 3\cdot5355 = \cdot5484508$, „ 123.

$\log 4\cdot6263 = \cdot6652338$, „ 94.

$L \cos 18° = 9\cdot9782132$.

CXXIX.

1. Define the cotangent of an angle, and trace the changes in the sign and magnitude of the cotangent as the angle varies from 0° to 360°.

Prove that $\cot a + \tan a = \operatorname{cosec} a \,.\, \sec a$.

2. Show in what cases $\sin\theta$ is less than $\operatorname{versin}\theta$, and find a general expression for all the angles $\sin^{-1}\dfrac{1}{\sqrt{2}}$.

3. Prove the formulæ—

$$(\alpha)\ \cot(A-B)=\frac{\cot A\ \cot B+1}{\cot B-\cot A}.$$

$$(\beta)\ \frac{1-\cos A}{1+\cos A}=\tan^2\frac{A}{2}.$$

$$(\gamma)\ \tan(45^\circ+A)-\tan(45^\circ-A)=2\tan 2A.$$

4. Prove that $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\dfrac{x+y}{1-xy}$.

If the lengths of the three straight lines OA, OB, OC be as $6:3:2$, and they be placed so that OB, OC are perpendicular to OA on opposite sides of it, then the angle BAC is equal to 45°.

5. If each of the angles of a triangle A and B is double of the third angle C, show that—

$$\cos\frac{A+B}{2}\cos\frac{A+B+C}{5}=\cos^4\frac{A+B+C}{4}.$$

6. Two sides and the included angle of a triangle being given, find the other angles and the third side.

Ex. $b=2{\cdot}25$ ft., $c=1{\cdot}75$ ft., $A=54^\circ$. Find B and C, having given—

$\log_{10}2 = {\cdot}301030$ $\qquad$ $L\tan 13^\circ\,47' = 9{\cdot}389724$

$L\cot 27^\circ = 10{\cdot}292834$ $\qquad$ $L\tan 13^\circ\,48' = 9{\cdot}390270.$

* 7. Explain how to calculate approximately the sine of 10″.

8. Show how to find the area of a quadrilateral by help of a measuring chain, but with no instrument for taking angles.

* 9. Explain the use of subsidiary angles. Show how to find by means of a subsidiary angle $\log (a+b)$, the logarithms of a and b being known.

10. Find the radius of the circle described so as to touch one side of a given triangle and the other sides produced.

* 11. In solving plane triangles show that there can be no ambiguity except when an angle has to be found by the sine (or cosecant), and never when the triangle is right-angled.

12. Find an expression for the area of a regular polygon of n sides inscribed in a circle of radius r; and thence deduce the area of the circle.

A given loop of string is formed into a number of regular polygons successively. Show that the polygon which has the greatest number of sides has the greatest area; and that the area of the triangle, square, and hexagon so formed are very nearly as 10 : 13 : 15.

WOOLWICH.

CXXX.

1. Explain the different methods of measuring an angle by means of degrees and circular measure, and write down the relations between these two measures of an angle.

Find the number of degrees, and the circular measure of the angle of a regular octagon.

2. Define the cotangent and cosecant of an angle, and prove by geometrical constructions the formulæ—

$$\operatorname{cosec}(180^\circ - A) = \operatorname{cosec} A, \quad \cot(180^\circ - A) = -\cot A,$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

3. Find a general expression for all the angles which have the same tangent as a given angle.

Solve the equation—

$$\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0.$$

* 4. If $A+B$ be less than a right angle, prove, by means of geometrical figures, the formulæ—

$$\sin(A+B)=\sin A\cos B+\cos A\sin B,$$

$$\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B).$$

5. Prove that—

$$\text{(i.)}\quad \tan\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.$$

$$\text{(ii.)}\quad \frac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=4\cos 2\theta\cos 4\theta.$$

$$\text{(iii.)}\quad \sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}.$$

6. Prove that—

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A},$$

and

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A},$$

and find the proper signs when the angle A lies between 990° and 1080°.

* Show, by a geometrical figure, that $\sin\frac{A}{2}$, when expressed in terms of $\sin A$, has four values.

7. Prove that in a triangle the sides are proportional to the sines of the opposite angles.

If, in a triangle ABC, $a\cos A=b\cos B$, prove that the triangle is either isosceles or right-angled.

8. The lengths of two of the sides of a triangle are 1 foot and $\sqrt{2}$ feet respectively, and the angle opposite to the shorter side is 30°. Prove that there are two triangles which satisfy these conditions, find their angles, and show that their areas are in the ratio $\sqrt{3}+1 : \sqrt{3}-1$.

9. Prove that, in a triangle ABC,

$$\cot\frac{C}{2}=\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}.$$

If $a=32$, $b=40$, $c=66$, find the angle C; given $\log 207=2{\cdot}3159703$, $\log 1073=3{\cdot}0305997$, $\text{L}\cot 66°\ 18'=9{\cdot}6424342$, and that the difference for 1′ is 3431.

10. A ship sailing due north observes two lighthouses bearing respectively N.E. and N.N.E. After sailing 20 miles the lighthouses are seen to be in a line due east. Having given—

$$\log 2 = \cdot 3010300 \qquad \log 11\cdot 715 = 1\cdot 0687423$$
$$\text{L}\tan 22^\circ\ 30' = 9\cdot 6172243 \qquad \log 11\cdot 716 = 1\cdot 0687794$$

find the distance in miles between the lighthouses accurately to four places of decimals.

CXXXI.

1. Define the *tangent* and *secant* of an angle, and obtain a formula connecting them. Deduce the corresponding formula connecting the cotangent and cosecant, and prove that—

$$(\tan\alpha + \operatorname{cosec}\beta)^2 - (\cot\beta - \sec\alpha)^2 = 2\tan\alpha\cot\beta\,(\operatorname{cosec}\alpha + \sec\beta).$$

2. Prove, by geometrical construction, that—

$$\sin A = \sin(180^\circ - A), \text{ and}$$
$$\cos(180^\circ + A) = \cos(180^\circ - A).$$

3. Express $\cos 2A$ in terms of $\cos A$ or $\sin A$, and show that—

$$8(\sin^4 A - \sin^2 A) + 1 = \cos 4A.$$

If $4A$ contains 40°, what four angles less than 180° will have their sines determined by this formula?

4. Prove that—

$$\text{(i.)}\ 2\cot 2A = \cot A - \tan A.$$
$$\text{(ii.)}\ \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{5}{13} = \sin^{-1}\frac{16}{65}.$$
$$\text{(iii.)}\ \cot(A + 15^\circ) - \tan(A - 15^\circ) = \frac{4\cos 2A}{2\sin 2A + 1}.$$

5. Find all the values of θ which satisfy the equation—

$$4\cos\theta - 3\sec\theta = 2\tan\theta.$$

6. Prove that the circular measure of an angle less than a right angle is intermediate in value between the sine and tangent of the angle.

Find approximately the number of minutes denoting the inclination to the horizon of an incline which rises $5\frac{1}{2}$ feet in 420 yards. $\left(\pi = \frac{22}{7}.\right)$

7. Show that in any plane triangle—

(i.) $a = b\cos C + c\cos B$.

(ii.) $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$.

8. Two adjacent sides of a parallelogram, 5 in. and 8 in. long respectively, include an angle of 60°. Find the lengths of the diagonals, and the area of the figure.

9. In the triangle ABC, $a=13$, $b=7$, $C=60°$. Find A and B, having given—

$\log 3 = \cdot 4771213$, $\quad L\tan 27°\,27' = 9\cdot 7155508$, $\quad$ diff $1' = 3087$.

10. A man observes that, when he has walked c ft. up an inclined plane, the angular depression of an object in the horizontal plane through the foot of the slope is α; and that, when he has walked a further distance of c ft., the angular depression of the same object is β. Show that the inclination of the slope to the horizon is $\cot^{-1}\{2\cot\beta - \cot\alpha\}$, and determine the distance of the object observed from the foot of the slope.

11. Find the radius of the circle inscribed in a triangle in terms of the lengths of the sides.

If d_1, d_2, d_3 be the distances of the centre of this circle from the angular points A, B, C respectively, prove that—

$$\frac{d_1^2}{bc} + \frac{d_2^2}{ca} + \frac{d_3^2}{ab} = 1.$$

ARMY.

CXXXII.

1. Define the circular measure of an angle, and find from your definition the circular measure of a right angle.

Express the angle 44° as a fraction of the angle 1·5 circular measure, assuming π to be $\frac{22}{7}$.

2. Prove that $\cos(A+B) = \cos A\cos B - \sin A\sin B$, and deduce the values of $\cos 2A$ and $\cos 3A$ in terms of $\cos A$.

Write down the values of $\sin 3A$ and $\tan 3A$.

3. Prove that $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$,

and that— $\sin A - \sin B = 2 \sin \frac{A-B}{2} \cos \frac{A+B}{2}$.

If $\sin A$ be the geometric mean between $\sin B$ and $\cos B$, prove that $\cos 2A = 2 \sin (45° - B) \cos (45° + B)$.

Prove also that—

$$2 \sin^{-1} \frac{1}{2} = \cos^{-1} \frac{1}{2}.$$

4. In the triangle ABC, if d be the diameter of the circumscribing circle, prove that—

(i.) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = d.$

(ii.) $\frac{b \cos C + c \cos B}{a} = \frac{c \cos A + a \cos C}{b} = \frac{a \cos B + b \cos A}{c} = 1.$

If D, E, F be the feet of the perpendiculars from A, B, and C upon the opposite sides, prove that the diameters of the circles circumscribing the triangles AEF, BDF, and CDE are respectively, $a \cot A$, $b \cot B$, and $c \cot C$.

* 5. In the triangle ABC, $A = 40°$, $a = 140{\cdot}5$, $b = 170{\cdot}6$; find B and C.

Given that $L \sin 40° = 9{\cdot}8080675$
$\log 1405 = 3{\cdot}1476763$
$\log 1706 = 3{\cdot}2319790$
$L \sin 51°\ 18' = 9{\cdot}8923342$
$L \sin 51°\ 19' = 9{\cdot}8924354$.

ANSWERS.

I.

1. 31·1943 . . . 2. $\pm\frac{3}{\sqrt{10}},\ \pm\frac{1}{\sqrt{2}}$. 3. $n\pi\pm\frac{\pi}{6}$.

6. $C=60^\circ\ 38'\ 57{\cdot}3''$, $\therefore\ A=81^\circ\ 45'\ 2{\cdot}7''$.

I.*

1. $\frac{\pi}{5}$.

6. 106 feet.

7. $c=1263{\cdot}619$.
$b=767{\cdot}878$.

3. (i.) $\theta=2n\pi\pm\frac{\pi}{2}$.

(ii.) $2\theta=n\pi$, or $\theta-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{4}$.

(iii.) $\theta=n\pi+\frac{\pi}{12}$.

II.

1. 60°, 36°, 84°.

$66\frac{2}{3}$, 40, $93\frac{1}{3}$ grades.

$\frac{\pi}{3}, \frac{\pi}{5}, \frac{7\pi}{15}$.

5. $B=79^\circ\ 6'\ 23{\cdot}4''$.
$C=40^\circ\ 53'\ 36{\cdot}6''$.

3. (i.) $2n\pi\pm\frac{\pi}{3}$, $n\pi$.

(ii.) $2\theta=(2n\pm1)\pi$.

(iii.) $\sin\theta=\sqrt{\frac{\sqrt{5}+3}{8}}$.

7. 18° 26′ 6″.

III.

1. $1+\frac{\pi}{36}=1{\cdot}0872$, $1-\frac{\pi}{36}={\cdot}9128$.

6. $\sin^{-1}\left\{\frac{\sin\gamma\cot a}{\sqrt{\cot^2 a+\cot^2\beta-2\cot a\cot\beta\cos\gamma}}\right\}$.

7. $A=36^\circ\ 52'\ 11{\cdot}6''$, $B=53^\circ\ 7'\ 48''$.

IV.

1. 435·908. 2. Negative. 6. $64^\circ\ 39'\ 22''$.

V.

1. $-2,\quad -\dfrac{2}{\sqrt{3}}$. 5. ·46929656.

$B=71^\circ\ 44'\ 29''$.

$C=48^\circ\ 15'\ 31''$.

6. $\dfrac{b}{\sqrt{2+2\sqrt{5}}}$.

VI.

2. $\sin 495^\circ=\dfrac{1}{\sqrt{2}},\quad \cos 495^\circ=-\dfrac{1}{\sqrt{2}}$. 3. $8\theta=2n\pi\pm2\theta,\quad x=\dfrac{1}{47}$.

$\theta=n\pi\pm\dfrac{\pi}{6}$, where $n=0, 1, 2$.

5. $4^\circ\ 55'\ 10{\cdot}6''$, $168^\circ\ 27'\ 25{\cdot}4''$.

VII.

1. ·3927 miles. 6. $61^\circ\ 17'\ 22''$.

7. $$\frac{a\sin\dfrac{\beta}{2}\cos\dfrac{a}{2}}{\sqrt{\cos^2\dfrac{\beta}{2}-\cos^2\dfrac{a}{2}}},\quad \frac{2a\sin\dfrac{a}{2}\sin\dfrac{\beta}{2}}{\sqrt{\cos^2\dfrac{\beta}{2}-\cos^2\dfrac{a}{2}}}.$$

VIII.

5. $55^\circ\ 46'\ 16''$. 6. $r_a=s\tan\dfrac{A}{2}$, etc. . . .

IX.

1. $2\theta=n\pi+(-1)^n\dfrac{\pi}{6}$. 4. $25\sqrt{2}(1+\sqrt{5})=114{\cdot}4$ feet.

5. $82^\circ\ 58'\ 45''$, $36^\circ\ 31'\ 15''$.

X.

1. $-1,\ 0,\ +1,\ +\sqrt{2},\ +1$. 4. 84.

5. $B=51^\circ\ 18'\ 21''$, or $128^\circ\ 41'\ 39''$. 7. 73·205 feet.

$C=88^\circ\ 41'\ 39''$, or $11^\circ\ 18'\ 21''$.

XI.

2. (i.) $4\theta = n\pi + (-1)^n\theta$, or $\frac{5\theta}{2} = 2n\pi \pm \frac{\pi}{2}$, $\frac{3\theta}{2} = n\pi$.

(ii.) $\theta = n\pi + (-1)^n \frac{\pi}{10}$, or $n\pi + (-1)^n \frac{13\pi}{10}$.

3. ·087.

8. 93° 0′ 16″, 26° 59′ 44″, $c = \sqrt{91} = 9{\cdot}54 \ldots$

XII.

1. 47° 44′ 47″ nearly.

3. $\sin 18° = {\cdot}309017$
$\sin\theta \;\; = {\cdot}414213$
$\sin 30° = {\cdot}5$.

4. $\theta = n\pi \pm \frac{\pi}{4}$, or $2n\pi \pm \frac{2\pi}{3}$.

6. 4° 55′ 10·6″, 168° 27′ 25·4″.

XIII.

1. 36°, 40^g, $\frac{\pi}{5}$.

4. $\frac{1}{8}(\sqrt{10+2\sqrt{5}} + \sqrt{3}(\sqrt{5}-1))$.

6. 108° 58′ 6·1″, 6° 1′ 53·9″.

8. $m^{\frac{2}{3}}n^{\frac{2}{3}}(m^{\frac{2}{3}} + n^{\frac{2}{3}}) = 1$.

XIV.

1. $\frac{4\pi}{3}$, 120°.

5. $\frac{\theta}{2} = n\pi + \frac{\pi}{4}$.

6. 75° 48′ 54″, 14° 11′ 6″.

7. 6 square feet.

XV.

1. 120°, $133\frac{1}{3}^g$, $\frac{2\pi}{3}$.

5. $2\theta = n\pi + \frac{\pi}{4}$.

6. 70° 33′ 30″, 19° 26′ 30″.

XVI.

1. $\frac{7\pi}{32}$, $43{\cdot}75^g$.

2. 1, 0, -1, $-\sqrt{2}$, -1.

5. $2A = 2n\pi \pm \left(\frac{\pi}{2} - A\right)$, or $\frac{\pi}{2} - 2A = n\pi + (-1)^n A$.

6. 109° 28′ 16″, 38° 56′ 34″, 31° 35′ 10″.

XVII.

1. $\frac{\pi}{3}, \frac{\pi^2}{180}, 45^\circ.$

3. $a \cos\phi, \; b \sin\phi, \; a \cos\phi_1, \; b \sin\phi_1.$

4. $C = 45^\circ$ or 135°, $B = 105^\circ$ or 15°, $b = \sqrt{3}+1$ or $\sqrt{3}-1$.

XVIII.

1. $\frac{3}{4}, \frac{3}{2\pi}.$

2. $\cot 2a, \quad \sec 2a.$

7. $90^\circ, \quad 210.$

8. $72^\circ \, 12' \, 59'', \quad 47^\circ \, 47' \, 1''.$

XIX.

3. $\frac{\sqrt{2+\sqrt{3}} - \sqrt{2-\sqrt{3}}}{\sqrt{2}} = \frac{1}{2}.$

4. $\sin^2 x = \frac{2ab+1 \pm \sqrt{1+4ab-4b^2}}{2(a^2+1)}.$

5. 8, 4.

6. $32^\circ \, 21' \, 54''.$

XX.

2. $\sin 18^\circ = \cdot 309 \ldots$
$\sin\theta \;\; = \cdot 414 \ldots$
$\sin 30^\circ = \cdot 5.$

4. $105^\circ, \quad 15^\circ.$

7. 229·6 yards.

XXI.

3. $n\pi + \frac{\pi}{4}.$

5. $42^\circ \;\; 0' \, 14''.$
$55^\circ \, 56' \, 46''.$
$82^\circ \;\; 3'.$

6. 102·91, 19·5379 sq. yards.

7. ·1612 sq. feet.

XXII.

1. $\frac{12}{13}, \; -\frac{12}{5}.$

5. $B = 1^\circ \, 1' \, 23''.$
$C = 147^\circ \, 28' \, 27''.$
$a = 4389 \cdot 717.$

6. 171·0836 yards.

XXIII.

1. 36°, 40°, $\frac{\pi}{5}$.

2. $\frac{4}{5}$, $\frac{3}{4}$, $\frac{5}{4}$.

6. 27° 30′, 56·369 feet.

7. B=54° 10′ 56″, or 125° 49′ 4″.
C=73° 30′ 4″, or 1° 51′ 56″.
c=196·8775, or 6·68.

XXIV.

1. $\frac{4\pi}{3}$.

2. -2.

7. 60° 52′ 29″, or 119° 7′ 31″.

8. 255·537.

9. 55° 46′ 16″.

XXV.

1. $\frac{3\pi}{5}$.

7. 3·849 yards.

8. 41° 24′ 34″.

9. 25·97, or 11·55 miles an hour.

XXVI.

5. $\frac{2ab}{a^2+b^2}$, $\frac{4ab\,(a^2-b^2)}{(a^2+b^2)^2}$.

8. B=51° 18′ 21″, or 128° 41′ 39″.
C=88° 41′ 39″, or 11° 18′ 21″.

9. B=49° 53′ 57·7″.
C=60° 6′ 2·3″.

XXVII.

3. $2\cos\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$.

4. C=53° 48′ or 126° 12′, A=91° 12′ or 18° 48′, a=81·4687 or 252·744.

5. $1 : 16\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}$.

6. $2\theta = n\pi + (-1^n)\frac{\pi}{2}$, or $\left(\theta - \frac{\pi}{4}\right) = 2n\pi$.

XXVIII.

1. 120^g, 108°. 2. 29·1478°, 28·1478°. 6. $\frac{b}{a}$.

8. 89·646 yards. 9. 74° 32′ 44″, 48° 59′ 16″.

XXIX.

1. $\frac{3\pi}{2(180-\pi)}$. 7. $B=41°\ 41'\ 28''$, or 138° 18′ 32″.

8. 100° 47′ 0·9″, 14° 12′ 59·1″.

XXX.

1. 1·0235 . . . 3. -1. 6. $A+\frac{\pi}{4}=2n\pi\pm\frac{\pi}{4}$.

9. 28·47717 square inches. 11. $\tan^{-1}\frac{a+b}{b\cot\alpha-a\cot\beta}$.

XXX.*

1. ·05585 . . ., 45° 50′ 11″, 32142·85 feet.

2. $A=2n\pi\pm\frac{\pi}{2}$, $B=n\pi$. 3. $\pi+\alpha$. 4. $n\pi+\frac{\pi}{4}$.

5. $A=75°$, $B=45°$. 6. $\frac{A}{2}$ lies between $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

7. -4, 1·8, $\bar{3}$·8326827, 1·09779. 9. 57·3979 inches.

11. $r=\frac{1}{2}(a+b+c)$, $\frac{r}{r_1}=\frac{(b+a)(a+b-c)}{(b-a)(a+c-b)}$.

XXXI.

1. 57° 16′ $21\frac{9}{11}$″, ·8675595. 3. $2n\pi+\frac{2\pi}{3}$. 5. 15°, 75°.

7. 2·30103, $\bar{2}$·39794, ·598626, 9·69897, 9·849435.

9. 40° 53′ 36″, 19° 6′ 24″, $c:a=\sqrt{7}:2$.

XXXII.

1. $\frac{\pi}{6}=\cdot 5236$, 9°. 2. 60°, $n\pi\pm\frac{\pi}{6}$. 4. $\frac{1}{\sqrt{3}}$, or $-\sqrt{3}$.

6. $2\cdot 0969100$, $\bar{4}\cdot 6989700$, $9\cdot 6989700$, $\cdot 4771213$.

9. $C=60°$ or $120°$, $A=90°$ or $30°$, $C=90°$, $100\sqrt{3}=173\cdot 205$.

10. $122\cdot 474487$. 11. $229\cdot 149$.

XXXIII.

1. 12·0031 inches. 2. $n\pi+(-1)^n\frac{\pi}{6}$, $n\pi\pm\frac{\pi}{4}$. 4. $\pm\frac{3}{4}$.

6. $5\cdot 6954904$, $\bar{4}\cdot 6954817$, $4\cdot 6954839$.

7. $C=18°$, $b=\sqrt{5}+1$, $a=\sqrt{5+2\sqrt{5}}+1$.

8. 78° 27′ 47″.

11. $200(\sqrt{3}-1)=146\cdot 41$, $200(\sqrt{3}+1)=546\cdot 41$.

XXXIV.

1. $135°-\frac{270°}{\pi}=49\cdot 056$, $\frac{3(\pi-6)}{4}=\cdot 85619$. 2. 120°, 240°.

3. $n\pi+\frac{\pi}{4}$, or $n\pi+\frac{\pi}{6}$. 7. $\cdot 69897$, $\bar{3}\cdot 90309$, $\bar{1}\cdot 45154$.

10. 74° 50′ 38″, 50° 32′ 58″. 11. 5 feet 4 inches, 6 feet $5\frac{1}{3}$ inches.

XXXV.

1. $17\frac{79}{88}=17\cdot 9049$. 3. $\sin C=\frac{63}{65}$, $\cos C=-\frac{16}{65}$.

6. $\theta=n\pi$, or $n\pi\pm\frac{\pi}{6}$. 8. $9\cdot 8494950$, $10\cdot 0624694$, $9\cdot 9849438$.

11. 35° 5′ 48″. 12. $2\frac{1}{2}$ miles.

XXXVI.

1. 57·2956, 2094·4. 5. $-\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{\sqrt{3}-1}{2\sqrt{2}}, \frac{1}{4}\sqrt{4-\sqrt{6}+\sqrt{2}}$.

6. $2n\pi\pm\frac{\pi}{6}$. 9. $9\cdot 427172$, 74° 29′ 12″, $\cdot 510544$.

10. $B=80°\ 46'\ 26\cdot 5''$, $C=63°\ 48'\ 33\cdot 5''$.

11. 20 ft., 40 ft.

XXXVII.

1. $18\frac{279}{331}°$, $58\frac{52}{331}°$. 8. $A=90°$, $B=30°$, $C=60°$.

9. 6·0389319, 9·6989700, 10·3010300.

10. $C=90°$. Case (β) is ambiguous. $C=38° 41'$, or $41° 19'$. 60·3894 feet.

XXXVIII.

1. 20 ft., 2600^g, $173\frac{1}{3}^g$. 2. $\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{3}}$.

3. $\frac{\sec A \sec B}{1-\sqrt{(\sec^2 A - 1)(\sec^2 B - 1)}}$.

5. 57·29577951° . . ., 3·1416, ·0002908882.

8. 115·47 feet, 100 feet.

9. $74° 13' 50''$, $35° 16' 10''$. 10. 114·412 feet.

11. $-2·3010300$, $\bar{3}·6989700$, 10·1030286, $38° 16' 15''$.

XXXIX.

1. 120^g, 108°. 3. $\cos(4A-B)\cos(3B-A)$.

6. 30°, 57·295° . . ., $\frac{\pi}{5}$. 7. 146·4 feet.

8. 90° or 30°, 60° or 120°, 200 or 100 feet.

9. ·54. 11. 9·9813707.

XL.

1. $8·\dot{3}'$. 2. 1, -1. 3. $\cos 4A - \cos 14A$.

4. $n\pi - \frac{\pi}{4}$. 6. 262·5 feet. 7. $26° 33' 55''$, $63° 26' 5''$.

8. 1034 feet. 11. $2\frac{2}{3}$ feet.

XLI.

1. 128°. 2. $\sin 2670° = \frac{1}{2}$. 10. 9·7565636.

XLII.

1. $30°, 60°, 90°; \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}; 33\frac{1}{3}^g, 66\frac{2}{3}^g, 100^g$.

2. $\frac{1}{\sqrt{2}}$. 6. $\frac{\theta}{2}=n\pi+\frac{\pi}{6}$. 10. 70° 53′ 40″, 49° 6′ 20″.

XLIII.

1. 1° 26′ nearly. 2. $\cos 165°=-\frac{\sqrt{3}+1}{2\sqrt{2}}$. 5. $n\pi\pm\frac{\pi}{3}$.

10. 33° 29′ 30″, 101° 30′ 30″.

XLIV.

1. $49\frac{1}{11}$ miles. 10. 2·529823. 11. 2·974 square feet.

XLV.

1. $20\frac{5}{11}°$. 2. 1 or $\frac{1}{2}$. 7. $n\pi+\frac{\pi}{4}$.

8. 105° 38′ 57″, 15° 38′ 57″. 12. ·555 square feet.

XLVI.

2. (i.) $\theta=n\pi\pm\frac{\pi}{4}$. (ii.) $4\theta=n\pi\pm\frac{\pi}{4}$. 3. $\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}$, 45°, 60°, 75°.

4. $\frac{7}{5\sqrt{2}}$. 7. 4028·5, 2031·7.

9. $c\tan\left(\frac{\pi}{4}-\frac{a}{2}\right)+h$, where h=given height.

XLVII.

1. 1·309 feet. 2. 0, 2, ∞, 0, 1. 11. 332.

XLVIII.

1. $\frac{8\pi}{9}$. 2. $-2, -\frac{2}{\sqrt{3}}$.

6. $x^2+y^2=a^2$. 10. $B=a$, $C=120°-a$.

XLIX.

2. Between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$; $\frac{\sqrt{3}}{2}$, $\sqrt{3}$, $-\frac{1}{\sqrt{2}}$.

6. 90°. 8. 1500 square feet. 9. 90°.

10. $3\frac{1}{3}$ miles per hour nearly. 11. 84 feet.

12. $5, 16, 5\sqrt{15}, 12$; 128° 39′ 13″, 51° 20′ 47″, 112° 23′ 37″, 67° 36′ 23″.

L.

2. 60°, 120°, 420°, $\frac{\sqrt{5}}{2}$. 8. 10, 4882240, 82° 45′ 22″.

9. $B=42° \; 17' \; 56''$, or $137° \; 42' \; 4''$; $C=101° \; 52' \; 4''$, or $6° \; 23' \; 56''$.

10. 474·333 feet.

LI.

1. $\frac{1}{60}$, $57\frac{3}{11}''$. 2. 0. 3. a.

4. $-\sqrt{2}$, $2-\sqrt{3}$, $-\frac{2}{\sqrt{3}}$, $-\sqrt{2}\,(\sqrt{3}+1)$.

6. $16\cos^5 A-20\cos^3 A+5\cos A$. 9. 92·788.

LII.

3. $2\sin 150°=\sqrt{1-\frac{\sqrt{3}}{2}}+\sqrt{1+\frac{\sqrt{3}}{2}}=1$

5. $\frac{5\theta}{2}=n\pi$, $\frac{7\theta}{2}=2n\pi\pm\frac{\pi}{3}$, $a^2\,\{y^2+(x+b)^2\}=\{y^2+x^2-b^2\}^2$.

10. 21° 23′.

LIII.

3. Trace $2\sin\left(\theta-\frac{\pi}{3}\right)$.

6. (i.) $\theta=n\pi$, or $2n\pi\pm\frac{\pi}{2}$. (ii.) $\frac{\theta}{2}=n\pi$, or $2n\pi$. (iii.) $x=y=2n\pi$.

7. $p^{\frac{2}{3}}\,q^{\frac{2}{3}}\,(p^{\frac{2}{3}}+q^{\frac{2}{3}})=1$.

10. Height$=b+(b-a)\,\frac{1-\cot\omega\,\tan a}{1-\cot\beta\,\tan a}$. 11. 81° 47′ 12″.

LIV.

1. 10, $\bar{8}$, $\bar{2}\cdot 90309$, $\cdot 09691$. 5. $\frac{3\pi}{5}$. 7. $A=90°$.

8. $B=54° \, 10' \, 55''$, or $125° \, 49' \, 5''$; $C=73° \, 30' \, 5''$, or $1° \, 51' \, 55''$.

9. 699·85484. 10. 3·65294. 11. $5\sqrt{10-2\sqrt{5}}$.

LV.

2. $64^{g} \, 1^{\backprime} \, 29\cdot\dot{6}2\dot{9}^{\backprime\backprime}$. 6. $B=36° \, 52' \, 11''$, $C=90°$, $c=95$.

7. $55° \, 46' \, 16''$. 8. $69° \, 10' \, 10''$, $46° \, 37' \, 50''$.

9. 5 miles.

LVI.

5. $\frac{\pi}{6}$, $42' \, 57\frac{3}{11}''$.

7. $A=53° \, 7' \, 42''$, $B=C=63° \, 26' \, 9''$, $b=c=394\cdot 364$.

8. 129·90375 square feet.

LVII.

6. $136° \, 35' \, 21\cdot 5''$, $13° \, 14' \, 33\cdot 5''$.

7. $86° \, 45' \, 56''$, or $33° \, 14' \, 14''$; diff. $=53° \, 31' \, 42''$.

LVIII.

1. $\frac{\pi}{2}+3\theta=2n\pi\pm 3\theta$; put $n=0$, 1, etc.

3. ·000000000000062208, 9·9375306.

5. 57295·84. 8. 14·1099 miles.

9. 17·3205 square chains. 10. $A=70° \, 31' \, 44''$.

LIX.

1. $111\frac{1}{9}^{g}$. 3. $2n\pi\pm\frac{\pi}{3}$.

6. $-\frac{\sqrt{2+\sqrt{2}}}{2}$. 9. $93° \, 41' \, 23''$, $26° \, 18' \, 37''$.

LX.

2. $\dfrac{1}{\sqrt{3}}$, $-\sqrt{3}$. 3. $-\dfrac{3}{\sqrt{2}}$, $\dfrac{5}{2}$, ∞, $-\dfrac{5}{2}$, $-\dfrac{3}{\sqrt{2}}$.

7. $\dfrac{c^2+d^2-a^2-b^2}{2ab}$. 12. 100 yards.

1. $\sqrt[6]{2}\left(\cos\dfrac{\pi}{12}-\sqrt{-1}\,\sin\dfrac{\pi}{12}\right)$, $\sqrt[6]{2}\left(\cos\dfrac{3\pi}{4}-\sqrt{-1}\,\sin\dfrac{3\pi}{4}\right)$,
$\sqrt[6]{2}\left(\cos\dfrac{17\pi}{12}-\sqrt{-1}\,\sin\dfrac{17\pi}{12}\right)$.

LXI.

1. $8\frac{4}{7}°$, $34\frac{2}{7}°$, $137\frac{1}{7}°$; or $25\frac{5}{7}°$, $51\frac{3}{7}°$, $102\frac{6}{7}°$.

3. $\sqrt{a^2+b^2}\sin(\theta+\phi)$, where $\tan\phi=\dfrac{a}{b}$; trace $2\sin\left(\dfrac{\pi}{3}+\theta\right)$.

5. $\sin a\,(\cos^2\gamma-\cos^2\beta)=\sin\beta\,(\cos^2\gamma-\cos^2 a)$. 9. $\sqrt{11}\,(6+2\sqrt{7})$.

LXII.

2. $-\dfrac{\sqrt{3}}{2}$, $-\sqrt{3}$, $\dfrac{1}{2}$; trace $\sqrt{2}\cos\left(\dfrac{\pi}{4}+A\right)$. 7. $\dfrac{13\pi}{15}$.

9. $C=93°\ 34'\ 58''$. Area$=70400$ square feet.

10. $B=71°\ 50'\ 53''$, or $108°\ 9'\ 7''$; $C=71°\ 9'\ 7''$, or $34°\ 50'\ 53''$;
4·97971, or 3·00664 miles an hour.

LXIII.

2. $-\dfrac{1}{2}$, $-\infty$; 120°, 240°, 480°. 5. $\dfrac{1}{4}\left(\sqrt{5+\sqrt{5}}-\sqrt{3-\sqrt{5}}\right)$.

6. $\dfrac{\pi}{8}$. 8. 12, 16, 20. 9. $A=33°\ 7'\ 22''$.

10. (1.) Impossible.
(2.) Ambiguous; $C=56°\ 28'\ 28''$, or $123°\ 31'\ 32''$.
(3.) $B=102°\ 59'\ 13''$, $C=33°\ 45'\ 47''$, $b=63·996$.

LXIV.

1. $\frac{\pi}{2} - 3\theta = 2n\pi \pm 2\theta$, where $n = 0, 1$, etc.

3. ·819918.

4. 55° 46′ 16″.

5. 53° 22′ 54″, or 126° 37′ 6″, 85° 37′ 6″, or 22° 22′ 54″.

6. $2s \sin \frac{A}{2}$.

7. $2^{\frac{1}{6}} \cos \frac{\pi}{12}$, $2^{\frac{1}{6}} \cos \frac{3\pi}{4}$, $2^{\frac{1}{6}} \cos \frac{17\pi}{12}$.

LXV.

1. Between $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

3. $3, \frac{7}{9}, \frac{1}{3}$.

4. 51° 18′ 21″, or 128° 41′ 39″; 88° 41′ 39″, or 11° 18′ 21″.

7. $y = \tan^{-1} \frac{B}{A}$, $2x = \log (A^2 + B^2)$.

LXVI.

6. (i.) $\cos \frac{(n+1)a}{2} \sin \frac{na}{2}$, (ii.) $\cot a - 2^n \cot 2^n a$.

LXVII.

2. $2 \sin \frac{A}{2} = -\sqrt{1 + \sin A} + \sqrt{1 - \sin A}$,

$2 \cos \frac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}$.

6. $B = 36°\ 52'\ 12''$, $C = 90°$, $A = 53°\ 7'\ 48''$.

7. $B = 65°\ 20'\ 3''$, or $114°\ 39'\ 57''$; $C = 92°\ 24'\ 57''$, or $43°\ 5'\ 3''$;
rate = 15·033107, or 21·98852.

LXVIII.

1. $\frac{4\pi}{9}$, 229·184° . . .

2. $11A = n\pi + (-1)^n 5A$, where $n = 0, 1$, etc.

4. $\sqrt{10 + 2\sqrt{5}} : 2\sqrt{10 - 2\sqrt{5}}$.

7. 142° 19′ 19″, 33° 40′ 41″.

LXIX.

1. $(n-1)\theta=m\pi,\quad (n+1)\,\theta=m\pi+(-1)^m\,\frac{\pi}{2}\,.$

3. $-\log\left(2\sin\frac{a}{2}\right)\cdot$ 4. $x=\pm 1\pm\sqrt{3}.$

LXX.

4. $(-1)^{n-1}\,\frac{1}{2}\tan\frac{\pi}{2n+1}\,;\ \frac{1}{2}\left(\tan^2\theta-\frac{1}{2}\tan^4\theta+\frac{1}{3}\tan^6\theta-\cdots\right)$

LXXI.

2. $-\frac{1}{2^9}\,(\cos 10\theta-10\cos 8\theta+45\cos 6\theta-120\cos 4\theta+210\cos 2\theta-126).$

LXXIV.

4. (i.) $\frac{1}{2}\,(\cos 2\theta-\cos 2^{n+1}\theta).$

(ii.) Apply $\cos^3 a=\frac{1}{4}\,(\cos 3a+3\cos a).$

$$\frac{\frac{3}{4}\cos\left(a+\frac{n-1}{2}\,\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}+\frac{\frac{1}{4}\cos\left(3a+\frac{n-1}{2}\,3\beta\right)\sin\frac{3n\beta}{2}}{\sin\frac{3\beta}{2}}.$$

LXXV.

5. (i.) $\dfrac{\sin\frac{(n+1)a}{2}\,\sin\frac{na}{2}}{\sin\frac{a}{2}}\,.$ (ii.) $e^{n\cos a}\sin\,(a+\sin a).$

LXXVI.

3. $B=\tan^{-1}2,\quad C=\tan^{-1}3,\quad a=3\sqrt{5},\quad b=6\sqrt{2},\quad c=9.$

LXXVII.

1. 1·1461280, 1·9912261, $\bar{1}$·727966.

LXXVIII.

1. $n\pi$, or $2n\pi \pm \frac{\pi}{2}$.

LXXIX.

5. $\frac{n \sin\theta \cos\theta + \cos(n+2)\theta \sin n\theta}{2 \sin\theta}$.

LXXX.

4. ± 3.

6. $\frac{\sin\frac{(n+1)}{2}a \sin\frac{na}{2}}{\sin\frac{a}{2}}$.

LXXXI.

1. $\frac{n\pi}{3}$, or $n\pi + \frac{\pi}{2}$, or $2n\pi \pm \cos^{-1}\sqrt{\frac{2}{3}}$.

4. $\frac{sabc}{2S}$.

3. $z^2(a+b-y-e)=(x-e)(a-y)(y-b)$.

LXXXII.

1. 4, -4, $\frac{3}{2}$; 18.

2. $\frac{r(2+\sqrt{3})}{\sqrt{3}}$.

5. $16x = n\pi + (-1)^n 8x$.

6. $\frac{\pi}{10}$, $\frac{3\pi}{10}$.

LXXXIII.

2. 1·991226, ·544068.

4. (i.) $b=c$. (ii.) $(b+c)=a\sqrt{2}$.

LXXXIV.

1. $\frac{1}{1+3ab}$, $\frac{3a}{1+3ab}$.

7. (i.) $\frac{\sin\frac{a+(n-1)(\pi+\beta)}{2}\sin\frac{n(\pi+\beta)}{2}}{\sin\frac{\pi+\beta}{2}}$.

(ii.) $\frac{\sin^{n+1}\beta \sin na - \sin^n\beta \sin(n+1)a + \sin a}{\sin^2\beta - 2\sin\beta \cos a + 1}$.

LXXXV.

7. (i.) $e^{-\cos\theta}\sin(\sin\theta)$. (ii.) $\frac{\pi^2}{6}$. (iii.) $\frac{\pi^2}{12}$.

LXXXVI.

8. (i.) $\dfrac{\sin\dfrac{(n+1)\theta}{2}\sin\dfrac{n\theta}{2}}{\sin\dfrac{\theta}{2}}$. (ii.) $\dfrac{\cos\dfrac{(n+1)\theta}{2}\sin\dfrac{\theta}{2}}{\sin\dfrac{n\theta}{2}}$.

LXXXVII.

1. $\bar{3}\cdot7481880$, $1\cdot5246090$.

3. $1,\ \frac{1}{4}(\sqrt{5}-1\pm\sqrt{-10-2\sqrt{5}}),\ \frac{1}{4}(-\sqrt{5}-1\pm\sqrt{2\sqrt{5}-10})$.

XCI.

4. $B=56^\circ\ 26'\ 33''$, or $123^\circ\ 33'\ 27''$; $C=93^\circ\ 33'\ 27''$, or $26^\circ\ 26'\ 33''$.
Rate $=2\cdot6718$.

XCII.

5. (i.) $\dfrac{\sin\dfrac{nA}{2}}{2\sin\dfrac{A}{2}}$. (ii.) $\dfrac{\sin A}{\dfrac{5}{4}-\cos A}$.

XCIII.

1. $2n\pi+\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$.

3. Area $=12$ square feet; $R=2\frac{1}{2}$ feet. 6. $\frac{1\pm\sqrt{-3}}{2}$, $\frac{-1\pm\sqrt{-3}}{2}$.

7. $\dfrac{\cos\left(a+\dfrac{(n-1)\beta}{2}\right)\sin\dfrac{n\beta}{2}}{\sin\dfrac{\beta}{2}}$, $\frac{1}{4}\{(2n+1)-\sin(2n+1)a\ \mathrm{cosec}\ a\}$.

XCV.

1. ·28854.

XCVI.

6. $\alpha-\theta=2n\pi$, $2\theta=n\pi$.

XCVII.

1. $\alpha+\theta=n\pi$, $\alpha-\theta=2n\pi$.
3. 75°, 45°.
6. 76° 47′ 2″, 49° 12′ 58″.

XCVIII.

1. 6.
3. $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $\frac{\sqrt{2+\sqrt{2}}}{2}$, $\sqrt{6}-\sqrt{3}+\sqrt{2}-2$.
4. $A=60°$, $B=45°$, $C=75°$.
8. $\dfrac{\sin\frac{n\beta}{2}\sin\left(\alpha+\frac{(n-1)\beta}{2}\right)}{\sin\frac{\beta}{2}}$,

$$\frac{c^{n-1}\sin(\alpha+(n-1\beta)-c^n\sin(\alpha+n\beta)-c\sin(\alpha-\beta)+\sin\alpha}{c^2-2c\cos\beta+1}.$$

XCIX.

1. $\frac{a^2+2ab+2b^2}{2b(a+b)}$, $\frac{a(a+2b)}{2b(a+b)}$.
6. (ii.) $\frac{1}{2}\left(e^{\sin\theta}+e^{-\sin\theta}\right)\cos(\cos\theta)$.

C.

1. $\frac{x^2}{3}+\frac{7x^4}{90}-\frac{13x^6}{405}$ . . . . . .
4. Solution impossible.
6. $\frac{(n+1)\sin n\theta-n\sin(n+1)\theta}{2(1-\cos\theta)}$.

CI.

1. $\tan\theta=\frac{2n+1}{4}\pm\frac{\sqrt{4n^2+4n-15}}{4}$.
3. $B=57° 35' 22''$, or $122° 24' 38''$; $C=87° 6' 9''$, or $22° 16' 53''$.
6. $z\sin\theta=\tan^{-1}\frac{b}{a}$, $z\cos\theta=\frac{1}{2}\log(a^2+b^2)$.

CII.

1. $\sin B = \frac{5}{8}$, ambiguous.

3. $A\frac{\sin(\theta+\phi)}{\cos\theta}$, where $\tan\phi = \frac{B}{A}$. 4. 1024.

CIII.

4. $c = 760{\cdot}39$. Area = 4 acres nearly.

6. (i.) $\frac{(n+1)\sin n\theta - n\sin(n+1)\theta}{2(1-\cos\theta)}$. (ii.) $-\log\left(2\sin\frac{\theta}{2}\right)$.

CIV.

3. The angles of the triangle formed by joining the centres of the escribed circles, viz. $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$, $\frac{\pi}{2}-\frac{C}{2}$.

4. $\frac{a\sin\alpha\sin\beta}{\sin\gamma}$.

CV.

1. 206265″.

7. $x = \tan^{-1}\frac{b}{a} + \cos^{-1}\frac{1}{2}\sqrt{a^2+b^2}$, $y = \tan^{-1}\frac{b}{a} - \cos^{-1}\frac{1}{2}\sqrt{a^2+b^2}$.

8. $x = \frac{1}{2}(a + \sqrt{a^2+2b^2})$, $y = \frac{1}{2}(\sqrt{a^2+2b^2} - a^2)$.

CVI.

1. $x = n\pi$, or $n\pi \pm \frac{\pi}{3}$. 3. $B = 117° 38' 45''$, $C = 27° 38' 45''$.

CVII.

5. $\frac{\cos\alpha - x\cos(\alpha-\beta)}{1 - 2x\cos\beta + x^2}$.

CIX.

1. 28° 38′ 11″. 5. −1. 6. 84.

CX.

1. $\dfrac{33}{56}$. 4. $B=90°$, $A=30°$.

6. (i.) $\dfrac{\sin\left(\alpha+\dfrac{(n-1)\beta}{2}\right)\sin\dfrac{n\beta}{2}}{\sin\dfrac{\beta}{2}}$. (ii.) $\dfrac{\sin\dfrac{n\alpha}{2}\cos\dfrac{(n+1)\alpha}{2}}{\sin\dfrac{\alpha}{2}}$.

CXI.

1. -4, -4.

4. $\dfrac{\sqrt{ab(a+b)}\ \sin\alpha\ \sin\beta\ \sin\gamma}{a\sin^2\alpha\,(\sin^2\beta-\sin^2\gamma)+b\sin^2\gamma\,(\sin^2\beta-\sin^2\alpha)}$.

a, b the distances between the points of observation.

CXIII.

1. $\cdot 000024240684\ldots$

CXIV.

1. 19. 3. $b^2=a^2+c^2-2ac\cos 2\alpha$.

CXV.

2. $\theta=n\pi\pm\dfrac{\pi}{3}$. 3. 3, $\bar{6}$. 6. $31\cdot 3968$.

7. ± 1, $\pm\sqrt{-1}$; $\dfrac{1\pm\sqrt{-3}}{2}$, $\dfrac{-1\pm\sqrt{-3}}{2}$.

CXVI.

2. $\dfrac{\pi}{2}+\theta=2n\pi\pm\alpha$, $\theta=2n\pi\pm\dfrac{\pi}{3}$. 3. $a\sin B=b\sin A$.

6. $\cos\dfrac{(2r+1)\pi}{6}+\sqrt{-1}\sin\dfrac{(2r+1)\pi}{6}$, $\pm\dfrac{\sqrt{3}\pm\sqrt{-1}}{2}$, $\pm\sqrt{-1}$.

CXVII.

1. 14, $\cdot 0512$.

CXVIII.

2. $\cot c=\dfrac{1}{a}-\dfrac{1}{b}$.

CXIX.

1. $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$

CXX.

1. $\frac{\theta}{2} = 2m\pi \pm \frac{\pi}{2}, \quad \frac{n\theta}{2} = m\pi + (-1)^m \frac{\theta}{2}.$

4. (ii.) $\cos \frac{n\pi}{2} \cos^n \frac{\pi}{2^n}.$ (iii.) $\frac{1 - 2a\cos\theta + a^2\cos 2\theta}{(1 - 2a\cos\theta + a^2)^2}.$

CXXI.

1. $\theta = n\pi + (-1)^n \frac{\pi}{6}, \quad \theta = n\pi \pm \frac{\pi}{6}.$ 4. 4° 55′ 10·6″, 168° 27′ 25·4″.

6. (i.) $\frac{1}{2}\sin n\theta \sin(n+2)\theta - \frac{1}{4}\sin 2n\theta \sin^2(n+2)\theta \operatorname{cosec} 2\theta.$

(ii.) $e^{c\cos\theta}\sin(c\sin\theta).$

CXXII.

2. 229·6473,
85·86.

8. $\dfrac{\sin\frac{(n+1)a}{2}\sin\frac{na}{2}}{\sin\frac{a}{2}}.$

CXXIII.

1. $0,\ 1 - \frac{1}{\sqrt{2}},\ -1,\ -\left(1 + \frac{1}{\sqrt{2}}\right),\ 0.$ 3. $2a.$

4. $B = 38° \ 12' \ 48''.$
$C = 21° \ 47' \ 12''.$

8. $\frac{2^{\frac{n}{2}}}{\underline{|n}} \cos\frac{n\pi}{4}.$

CXXIV.

3. $\theta = \tan^{-1}\left\{\frac{n-m}{n+m}\cot a\right\}.$ 4. Solution impossible.

CXXV.

2. 54°. 3. $B=117°\,38'\,45''$, $C=27°\,38'\,45''$.

4. (i.) $\dfrac{\sin\dfrac{(n+1)\,\theta}{2}\sin\dfrac{n\theta}{2}}{\sin\dfrac{\theta}{2}}$. (ii.) $\dfrac{\sin\theta}{1+\sin^2\theta-\sin 2\theta}$.

CXXVI.

1. $x=\dfrac{2\,(\pi+a)}{5}$, $y=-\dfrac{\pi+6a}{5}$. 2. 255·53863.

5. $e^{x\cos a}\sin(x\sin a)$.

CXXVII.

1. 31° 13′ 0·696″, 1·026366. 6. $A=105°$, $B=45°$.

8. 120°. 10. $300\sqrt{2}=141{\cdot}4213$ yards.

11. 3, 5, 7; 120°.

CXXVIII.

3. $\dfrac{1}{\sqrt{2}}$, $\dfrac{1+\sqrt{5}}{4}$, 9·8494850, $\theta=n\pi+(-1)^n\dfrac{\pi}{6}$.

4. $2\sin\dfrac{A}{2}=-\sqrt{1+\sin A}-\sqrt{1-\sin A}$.

6. 46·2636 yards from *B*. 7. 142° 19′ 19″, 33° 40′ 41″.

CXXIX.

2. $n\pi+(-1)^n\dfrac{\pi}{4}$. 6. 76° 47′ 2″, 49° 12′ 58″.

CXXX.

1. 135°, $\dfrac{3\pi}{4}$. 3. $n\pi+\dfrac{\pi}{4}$, or $n\pi+\dfrac{\pi}{3}$.

6. First equation $-$, Second equation $+$.

8. 45° or 135°, 105° or 15°. 9. 132° 34′ 32″. 10. 11·7157.

CXXXI.

3. $\pm 10°$, $\pm 80°$. 5. $n\pi + (-1)^n \frac{\pi}{10}$. 6. $15'$.

8. 7, $\sqrt{129} = 11{\cdot}3578167\ldots$, area $= 20\sqrt{3} = 34{\cdot}641\ldots$

9. $A = 87°\ 27'\ 25''$, $B = 32°\ 32'\ 35''$.

CXXXII.

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